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Do Skeletal Networks Derived from Water Bodies Follow Horton's Laws?[1]

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The aim of this short note is to test whether the morphological skeletal network (MSN) *of water bodies that resembles a river network follows Horton's laws. A fractal relationship of* MSN *of a water body is also shown. This investigation shows that the* MSN *of the Nizamsagar reservoir follows Horton's laws. Furthermore, this reservoir has a fractal dimension (D_m) of 1.92 which was computed by using two morphometric quantities and the fractal dimension of the main skeletal length* (d). *This value tallies exactly with the fractal dimension (D_f) of the whole* MSN *computed through box-counting method.*

KEY WORDS: mathematical morphology, structuring element, morphometric quantities, fractal.

INTRODUCTION

Morphological skeleton network (*MSN*) which is a one pixel wide caricature that summarizes overall shape, size, orientation, and connectivity of a feature (e.g., lakes, river basin outlines), is an example of a fractal tree (Sagar, 1996). The structural or topological similarities between the *MSN* of a water body and natural river network induced the authors for this study. These similarities lead to verify the Horton's laws (Horton, 1945) on *MSN*, and also to show a fractal relationship of *MSN*. The organisation of this letter is as follows. In what follows in this short note is a brief introduction to certain mathematical morphological transformations that are needed to extract the morphological skeleton of an object, description to Horton's laws of number and length of streams, fractal nature of the structure of the water body, and a sample study to verify the Horton's laws of river networks with the *MSN* of a water body.

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MATHEMATICAL MORPHOLOGY

Mathematical morphology (Serra, 1982) based on set theoretic concepts is one of the approaches to the analysis of geometric properties of different structures. A discrete binary image *A* is defined as a finite subset of Euclidean two-dimensional space, Z^2. The geometrical properties of a binary image possessing set (*A*) and set complement (A^c) are subjected to the morphological functions. The morphological operators can be visualised as working with two images. The image being processed is referred to as the binary image *A*, and the other image being a structuring template, *B*. Each structuring template possesses a designed shape that can be thought of as a probe of *A*. In addition, many structuring templates are represented by a compact subset of Euclidean space, so that constraints which correspond to the four principles of the theory of mathematical morphology such as invariance under translation, compatibility with change of scale, local knowledge, and upper-semicontinuity will be imposed on morphological set transformations such as erosion, dilation, opening, and closing for precise extraction of topological information. The transformations involved in *MSN* extraction are erosion to shrink, dilation to expand, and cascade of erosion-dilation to smoothen the set. These transformations are based on Minkowski set addition and subtraction (Serra, 1982).

Dilation

Dilation combines two sets using vector addition of set elements. If *A* and *B* are sets in Euclidean space with elements *a* and *b*, respectively, $a = (a_1, \ldots, a_N)$ and $b = (b_1, \ldots, b_N)$ being *N*-tuples of element coordinates, then the dilation of *A* by *B* is the set of all possible vector sums of pairs of elements, one coming from *A* and the other from *B*. The dilation of *A* with structuring template, *B*, is defined as the set of all points "*a*" such that B_a intersects *A*. It is expressed as

$$A \oplus B = \{a : B_a \cap A \neq \emptyset\} = U_{b \in B} A_b \tag{1}$$

Erosion

The erosion of *A* with structuring template, *B*, is defined as the set of points "*a*" such that the translated B_a is contained in *A*. It is expressed as

$$A \ominus B = \{a : B_a \subseteq A\} = \cap_{b \in B} A_b \tag{2}$$

where $B = \{b : b \in B\}$, i.e., *B* rotated 180° around the origin. It is important to mention here that Minkowski addition and subtraction are akin to the morphological dilation and erosion respectively as long as the structuring template (*B*) is of symmetric type (Maragos and Schafer, 1986). The mathematical representation of the decomposition of the structuring template of size B_n (where $n = 0, 1, 2, \ldots, N$) into smaller structuring templates can be shown as

$$B_n = B \oplus B \oplus B \oplus \ldots \oplus B \tag{3}$$

According to Equation (3), two consecutive erosions and dilations can be, respectively, represented as $(A \ominus B) \ominus B = A \ominus B_2$, and $(A \oplus B) \oplus B = A \oplus B_2$. The dilation followed by erosion and the erosion followed by dilation are termed as closing and opening transformations, respectively. These cascade transformations are idempotent (Serra, 1982). However, these transformations can be performed according to the multiscale approach (Maragos and Schafer, 1986). In the multiscale approach, the size of the structuring template is increased from iteration to iteration.

EXTRACTION OF MORPHOLOGICAL SKELETON NETWORK

A morphological skeleton network is defined as a line thinned caricature to summarize the shape, size, orientation, and connectivity of the object (Lantuejoul, 1980). The object refers to water body in the present context. *MSN* is defined according to (5) as the union of maximum possible angular points that could be isolated from all possible successive front lines of the water body that is represented in binary form. The possible skeletal network from the image (A), $MSN(A)$, viewed as subsets of Z^2 can be defined mathematically as

$$MSN_n(A) = (A \ominus B_n)/\{[(A \ominus B_n) \ominus B] \oplus B\} \qquad n = 0, 1, 2, \ldots, N \tag{4}$$

$$MSN(A) = \mathbf{U}_{n=0}^{N} MSN_n(A) \tag{5}$$

where $MSN_n(A)$ denotes the nth skeletal subset of the image (A). In the above expression, subtracting from the eroded versions of A their opening by B retains only the angular points, which are points of the skeletal network. The union of all such possible points produces a skeletal network $MSN(A)$ of image A. The subscript, n ranges between 0 and N, and is the size of the structuring template. In Equation (4), opening of an eroded set is always by means of a structuring template of an arbitrary size.

HORTONIAN LAWS OF RIVER NETWORKS

On the basis of the intuitive arguments it is hypothesised that the morphological skeleton network extracted from a water body with respect to fractal structure follows certain laws of river morphometry (Sagar, 1996), proposed by Horton (1945) and Schumm (1956). To test this hypothesis, it is imperative here to describe Horton's (1945) laws of river networks.

In the geomorphological analysis of river networks, scaling properties are defined by Horton's laws (Smart, 1972). These laws are supplemented by Strahler's (1964) ordering (e.g., Shreve, 1967; Smart, 1972) to yield stream bifurcation number and stream length laws. These laws of channel network composition are an integral part of Strahler's (1957) ordering technique. A system of ordering that recognizes the existence of hierarchy among the separate segments is therefore assumed to represent the structure of channel networks. This postulates that source channels are of order 1, and when two channels of order i and j, respectively, merge, a channel of order ω is formed as

$$\omega = \max\{i, j, \mathrm{Int}[1 + (1/2)(i + j)]\} \tag{6}$$

where function Int [ ] denotes the integer part of the argument.

According to Equation (6), when two channels of equal order join, a stream of one order higher is formed; and when two streams of different order join, the continuing channel retains the order of the higher-order channel.

Horton's laws of stream numbers, and stream length propose that the bifurcation ratio (R_B) and the stream length ratio (R_L) are constant over homogeneous river basins at increasing resolution (Horton, 1945). These ratios can be computed as follows: Bifurcation ratio (R_B) is the ratio of number of stream segments of a given order $N(\omega - 1, \Omega)$ to the number of streams of the next higher order, $N(\omega, \Omega)$

$$R_B = \frac{N(\omega - 1, \Omega)}{N(\omega, \Omega)} \qquad \omega = 2, \ldots, \Omega \tag{7}$$

Stream length ratio (R_L) is the ratio of mean length of segments of order $\omega, \bar{L}(\omega, \Omega)$ and mean length of segments of the next lower order, $\bar{L}(\omega - 1, \Omega)$

$$R_L = \frac{\bar{L}(\omega, \Omega)}{\bar{L}(\omega - 1, \Omega)} \qquad \omega = 2, \ldots, \Omega \tag{8}$$

where $N(\omega, \Omega)$ is the number of streams in order ω in a basin of order Ω, $(N(\Omega, \Omega) = 1)$; $\bar{L}(\omega, \Omega)$ is the average length of streams of order ω; R_B and R_L can also be obtained from the slopes of the straight lines resulting from plots of logarithmically transformed values of $N(\omega, \Omega)$ and $L(\omega, \Omega)$ vs. order ω, for ω ranging from 1 to Ω. Channel network density: The channel density C_d can be calculated by

$$C_d = L(\Omega)/A(\Omega) \tag{9}$$

where $A(\Omega)$ and $L(\Omega)$ are the finite measures of the basin area and the length of total channel network (i.e., scale dependent). Channel frequency: Channel frequency, C_f, can be calculated by

$$C_f = \sum_{\omega=1}^{\Omega} N(\omega, \Omega)/A(\Omega) \tag{10}$$

MELTON'S LAW

Melton's law (Smart, 1972) relates the channel frequency (10) to the square of channel density (9).

$$\frac{\sum_{\omega=1}^{\Omega} N(\omega,\Omega)/A(\Omega)}{[L(\Omega)/A(\Omega)]^2} = \frac{C_f}{C_d^2} \cong 0.69 \qquad (11)$$

The ratio of these two quantities has a value of about 0.69. There are two proposed methods to test whether a model is Hortonian or non-Hortonian. The former is based on Melton's law and the later is to examine the channel density across scales. In a non-Hortonian system, the channel density does not vary with scale; in a Horton system it does (Beer and Borgas, 1993). Melton's law (Smart, 1972) may be computed to test how closely a Hortonian stream network obeys Melton's law. These laws proposed in the context of channel network are used to study *MSN* of a water body.

DO SKELETAL NETWORKS OF WATER BODIES FOLLOW HORTON'S LAWS?

What follows in this section is a brief description on the structure of the water body and its morphological skeletal network and the fractal nature of water body outlines. A sample study on the Nizamsagar reservoir is shown to test the Horton's laws and a fractal relationship proposed in the context of river networks.

Structure of Morphological Skeleton Network of Water Bodies

Any planar feature can be transformed as its morphological skeleton. As a river network is assumed as the morphological skeleton of a basin, the *MSN* of water bodies is assumed as *fluid flow attracting points*. As in a river basin where the pattern of stream branches is determined by inequalities in the outline of the basin, the morphological skeleton network also will be determined by the inequalities in the outline of the water body. These inequalities are caused by several conditions such as climatic, environmental, physiographic, geological etc. These wrinkles can be treated as the source paths of skeletal branches. The complexity of morphological skeleton depends upon the degree of contortions in the basin outline. In a similar way, a complex morphological skeleton network will be produced due to inequalities in the outline of the water body. The number of branching orders depends upon the overall structure and textural details of the water body. The higher the textural information the more is the number of lower order skeletal segments. Hence, to characterize a water body, both structural and textural aspects must be studied. Depending upon the overall structural composition of a water body, the skeletal branches of lower order bifurcate and skeletal branches of next higher order are formed. This implies that the number of wrinkles in the boundary of the water body is equal to the number of first-order skeletal branches. The two first orders join to form second order and so on, as in the case of river networks.

Fractal Nature of Water Body Outline

While characterizing and quantifying surface water bodies, variations in their number and shape naturally occur because of the different scales used. These variations in number and shape of water bodies from scale to scale lead to many ambiguities. Thus, scale can have profound implication on the quantification of water bodies. At lower resolutions, the boundary of surface water bodies simulates a smooth curve. Thus water bodies also exhibit fractal characteristics and statistical self-similarity according to the concept of Mandelbrot (1982). The outline of surface water bodies also exhibits convolutions with increasing magnification. At higher resolutions, the smaller edge effects are apparent in the boundaries. Hence, with increasing resolution the order of the morphological skeleton network, determined by inequalities in the outline, will be increased as in the case of river networks. However, the skeletal density changes with change in resolution. Such a change in river networks enables the river network to be Hortonian. Because both water body outline and the corresponding *MSN* possess statistical self-similarity, they are fractals. Hence, an established method to compute fractal dimension using morphometric order ratios (Tarbotan, Bras, and Rodriguez-Iturbe, 1990) is adopted in the sample study to arrive at the fractal dimension for the *MSN* of the Nizamsagar reservoir.

A SAMPLE STUDY AND CONCLUSIONS

To test whether the water body follows Horton's laws an image of the Nizamsagar reservoir situated in Andhra Pradesh, India, acquired on 13-04-1989 from IRS 1A, LISS 1 sensor with a resolution of 72 m (Fig. 1) is considered. The *MSN* (Fig. 2) is extracted according to the procedures detailed in Eqs. (4) and (5). Strahler's (1964) order designation is given to the morphological skeleton network thus extracted. At this resolution of the image, the network possesses four orders. Hence, it is termed as fourth order skeletal network ($\Omega = 4$). The basic measures such as order-wise lengths, total length, and main length of skeletal network are computed (Table 1). The morphometric order ratios are computed by following Eqs. (7) and (8). Table 1 shows these results.

Figure 1. The extracted Nizamsagar reservoir from IRS 1A LISS 1, 72 m resolution.

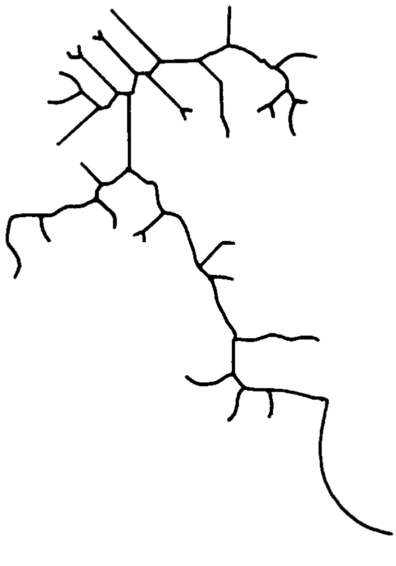

Figure 2. The morphological skeleton network of the Nizamsagar reservoir.

Table 1. Basic Measures of Morphological Skeleton Network, Certain Morphometric Order Ratios, and Dissection Properties

No. of skeletal orders				Length of skeletal orders in pixel units										
1	2	3	4	1	2	3	4	R_L	R_B	C_d	C_f	C_f/C_d^2	D_m	D_f
28	7	2	1	356	96	76	200	2.16	3.33	0.06	0.0025	0.62	1.91 2	1.92

Figure 3A and B show a linear relationship between order of the skeletal branches with logarithms of number of skeletal segments and mean lengths of skeletal branches of each order. It is deduced that from the statistical relations shown in Figure 3A and B, the *MSN* of the Nizamsagar reservoir follows Horton's laws of river networks. It is tested whether the skeletal density (ratio of the length of skeletal segments and the area of the water body) varies with increasing resolution of the image. The other way to test the Hortonity of the network is to verify Melton's law (11), which is 0.62 for the *MSN* of the Nizamsagar reservoir, close to 0.69 in the context of river networks. This shows that the skeleton network of this water body is Hortonian. It is also observed that this *MSN* follows a fractal relationship proposed in the context of river morphometry. The morphological skeleton network of the Nizamsagar reservoir follows Horton's laws as evidenced by the results. The fractal dimension (D_m) is computed through Equation (12) proposed by Tarbotan, Bras, and Rodriguez-Iturbe (1990) in the context of river networks. The fractal dimension of such a fractal tree can be computed by the following standard ordering quantities initially proposed in the context of river morphometry. These include bifurcation ratio, length ratio, and the fractal dimension of the main skeletal length (d) of the *MSN* of the Nizamsagar reservoir to compute the fractal dimension (D_m).

$$D_m = d \, \frac{\log(R_B)}{\log(R_L)} \tag{12}$$

In Equation (12), the fractal dimension of the main skeletal length of the water body (d) computed by box counting method (Feder, 1988) is 1.21, and R_B, and R_L are the bifurcation and length ratios of *MSN*. The fractal dimension (D_m) computed for the *MSN* of Nizamsagar reservoir through Equation (12) is 1.912. This value tallies with the fractal dimension (D_f) of the *MSN* computed by box counting method, i.e., 1.92. The fractal plot to compute fractal dimension (D_f) through box-counting method is shown in Figure 4.

It is inferred that the morphological skeleton of the Nizamsagar reservoir follows Horton's laws and also the fractal relation proposed in the context of

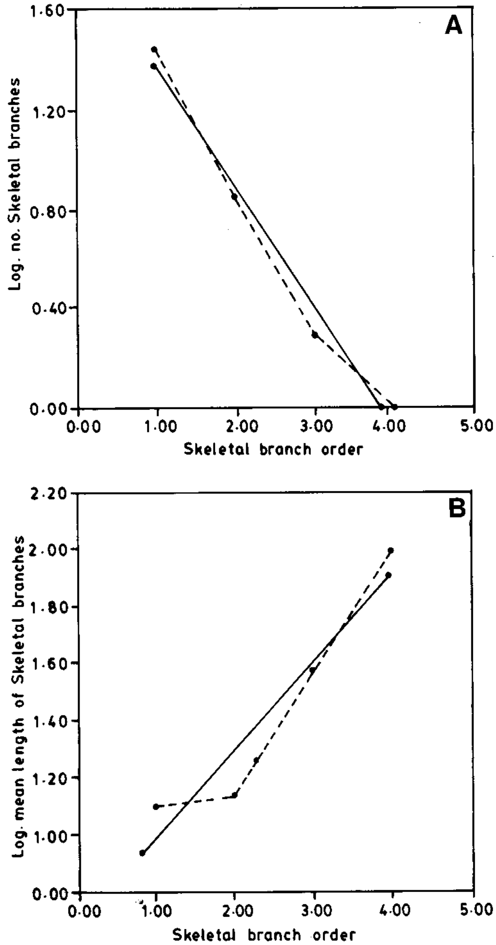

Figure 3. Statistical results of morphological skeleton network of Nizamsagar reservoir. A, The log of the number of skeletal segments of a given order plotted against that order. B, The log of the average length of skeletal segments of a given order plotted against that order. Horton's laws state that a natural river network yields a linear relation on each graph.

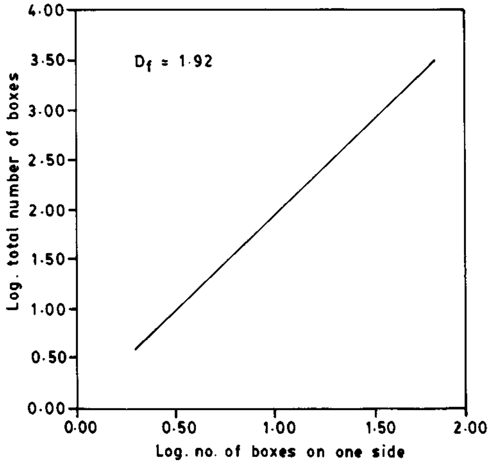

Figure 4. A fractal plot between log no. of boxes on one side and log number of boxes through which the skeleton network traverses.

river network. The fractal dimension (D_m) tallies with the fractal dimension (D_f) of the MSN of the Nizamsagar reservoir. This similarity supports that the MSN of Nizamsagar reservoir follows Horton's laws.

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APPENDIX. LIST OF SYMBOLS AND NOTATIONS

Z^2,	Two-dimensional grid of discrete space
$A, B, M, \ldots$,	Subsets of Z^2
$\emptyset$,	Empty set
$a, b, m, \ldots$,	Elements of vector points Z^2, i.e., a point in the 2-D space
$a \in A$,	Element "a" belongs to A
$a \notin A$,	Element "a" does not belong to A
$\subset, \subseteq, \cap$, U,	Subset, improper subset, intersection, and union
$A \ominus B$,	Minkowski subtraction
$A \oplus B$,	Minkowski addition
$A \ominus \hat{B}$,	Erosion of A by B
$A \oplus \hat{B}$,	Dilation of A by B
M/A,	Set difference between M and A
A^c,	Complement of A with respect to Z^2
A_b,	Translate of "a" by vector b, i.e. $\{a : a - b \in A\}$
$\cap_{b \in B} A_b$,	Intersection of all the translates of A_b, with $b \in B$
$U_{b \in B} A_b$,	Union of all the translates A_b, with $b \in B$
Ω,	Order of the channel network in the basin
$N(\omega, \Omega)$,	Number of channel segments of order ω within a network of order Ω
$L(\omega, \Omega)$,	Channel length of order ω within a network of order Ω
$N(\Omega)$,	Total number of channels within a network of order Ω
$L(\omega, \Omega)/N(\omega, \Omega)$,	Mean length of channel segments of order ω
$L(\Omega)$,	Total length of the channel network in the basin of order Ω

D_f, Fractal dimension computed through box-counting method
D_m, Fractal dimension computed through morphometric order ratios
d, Fractal dimension of main skeletal length computed through box-counting method.