1. (a) Let $H$ be a subgroup of $G$, and $a \in G$. Let $aHa^{-1} = \{aha^{-1} | h \in H\}$. Show that $aHa^{-1}$ is a subgroup of $G$.
   (b) If $H$ is finite, what is $o(aHa^{-1})$?
2. Prove that every subgroup of a cyclic group is itself cyclic.
3. Consider the subgroup $H = n\mathbb{Z}$ of $\mathbb{Z}$. Find all left cosets of $H$ in $\mathbb{Z}$. Determine the index of $H$ in $\mathbb{Z}$.
4. Prove that every subgroup of an abelian group is normal.
5. Prove that every subgroup of index 2 is normal.
6. Prove that a subgroup $H$ of a group $G$ is a normal subgroup if and only if it is the kernel of a homomorphism.
7. Let $G$ be a finite group of order 21 and let $K$ be a finite group of order 49. Suppose that $G$ does not have a normal subgroup of order 3. Then determine all group homomorphisms from $G$ to $K$.
8. Suppose $H$ is the only subgroup of order $o(H)$ in a finite group $G$. Prove that $H$ is a normal subgroup of $G$.
9. If $G$ is abelian and if $N$ is any subgroup of $G$, prove that the quotient group $G/N$ is abelian.
10. Show that a group $G$ is cyclic if and only if there exists a surjective group homomorphism from the additive group $\mathbb{Z}$ of integers to the group $G$.
11. (**Practice problem**) Consider the additive quotient group $\mathbb{Q}/\mathbb{Z}$.
    (a) Show that every coset of $\mathbb{Z}$ in $\mathbb{Q}$ contains exactly one representative $q \in \mathbb{Q}$ in the range $0 \leq q < 1$.
    (b) Show that every element of $\mathbb{Q}/\mathbb{Z}$ has finite order but that there are elements of arbitrarily large order.
    (c) Show that $\mathbb{Q}/\mathbb{Z}$ is the torsion subgroup of $\mathbb{R}/\mathbb{Z}$.
    (d) Prove that $\mathbb{Q}/\mathbb{Z}$ is isomorphic to the multiplicative group of roots of unity in $\mathbb{C}$.