1. Consider $\{y_n\}_{n=1}^{\infty}$given by
$$y_n = \sum_{k=0}^{n} \frac{1}{k!}$$
for all $n \in \mathbb{N}$. Using the comparison test appropriately show that $\{y_n\}_{n=1}^{\infty}$converges to a number between 2 and 3. *We call this limiting number e.*

2. Suppose $\{z_n\}_{n=1}^{\infty}$ is a sequence of real numbers. Decide whether the series $\sum_{n=0}^{\infty} z_n$ converges in each of the following cases:

   (a) $x \in \mathbb{R}$ and for $n \geq 0$, $z_n = nx^n$

   (b) $x \in \mathbb{R}$ and for $n \geq 0$, $z_n = \frac{x^n}{n!}$

   (c) $x \in \mathbb{R}$ and for $n \geq 0$, $z_n = \frac{(x+5)^n}{n(n+1)}$

3. (*Exponential function : e^x*) Consider the function $E : \mathbb{R} \to \mathbb{R}$ given by
$$E(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}.$$

   (a) Show that E is well-defined and $E(x+y) = E(x)E(y)$, for all $x, y \in \mathbb{R}$.

   (b) Show that E is a continuous and monotonically increasing (strictly) function on $\mathbb{R}$.

   (c) Let $e = E(1)$. Show that $E(x) = e^x$ for all $x \in \mathbb{R}$.

   (d) Show that $\lim_{x\to\infty} x^n e^{-x} = 0$ for all $n \in \mathbb{N}$.

1. Let y_n and e be as in Problem 1 of Worksheet 2. Show that $0 < e - y_n < \frac{1}{n!n}$ and conclude that e is not rational.

2. Suppose $\{z_n\}_{n=1}^{\infty}$ is a sequence of real numbers. Decide whether the series $\sum_{n=0}^{\infty} z_n$ converges in each of the following cases:

   (a) $z_n = \frac{n!}{n^n} x^n$

   (b) $z_n = \frac{(n!)^2}{n^n} x^n$

   (c) $z_n = \frac{n!}{n^n} (x-2)^n$

3. Let $\{a_n\}_{n \geq 1}$ be a sequence of numbers such $0 \leq a_n < 1$.

   (a) Using induction, show that $\prod_{i=1}^{n} (1 - a_i) \geq 1 - \sum_{k=1}^{n} a_k$.

   (b) Show that $1 - a \leq e^{-a}$ for any $a \in [0, 1)$.

   (c) For any $n \geq 1$, let $b_n = \prod_{i=1}^{n} (1 - a_i)$.

      i. Show that b_n converges to 0 if $\sum_{k=1}^{\infty} a_k = \infty$.

      ii. Show that b_n converges to $b \in (0, 1)$ if $\sum_{k=1}^{\infty} a_k < \infty$.