June 18th, 2019 **SWMS-2018-Follow up** **Worksheet 1**

Discrete Time Fourier Transform for infinite sequences

Notation: For $a, t \in \mathbb{Z}$

$$\delta_a(t) = \begin{cases} 1 & \text{if } t = a \\ 0 & \text{otherwise} \end{cases}$$

$$u_t = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Let $\{ \quad a_t \quad : \quad t = \ldots, -1, 0, 1, \ldots\}$ be an infinite sequence of real (or complex numbers) such that

$$\sum_{t=-\infty}^{\infty} \mid a_t \mid^2 < \infty.$$

Then the Discrete Time Fourier Transform (DTFT) of a is given by

$$A(f) \equiv \sum_{t=-\infty}^{\infty} a_t e^{-i2\pi f t}, \qquad -\infty < f < \infty$$

1. Determine the Fourier transform of each of the sequences below:
   (a) $x_t = \delta_{-3}(t)$ for all $t \in \mathbb{Z}$.
   (b) $x_t = \frac{1}{2}\delta_{-1}(t) + \delta_0(t) + \frac{1}{2}\delta_1(t)$ for all $t \in \mathbb{Z}$.
   (c) $x_t = u_{t+3} - u_{t-4}$ for all $t \in \mathbb{Z}$.

2. Let $\{x_t\}_{t\in\mathbb{Z}}$ and $\{X(f)\}_{-\frac{1}{2}<f<\frac{1}{2}}$ be a bi -infinite sequence and its DTFT. Determine in terms of X the Fourier transform of each of the following sequences:
   (a) $y_t = kx_t$, for all $t \in \mathbb{Z}$, with $k \in \mathbb{C}$.‘
   (b) $y_t = x_{t-t_0}$ for all $t \in \mathbb{Z}$, with $t_0 \in \mathbb{Z}$

3. Consider the following bi-infinite sequences given by

   $$a_t = \alpha^t u_t \text{ and } b_t = \beta^t u_t$$

   for all $t \in \mathbb{Z}$, with $0 < \alpha < 1$ and $\beta \in \mathbb{C}$, $0 <\mid \beta \mid< 1$.
   (a) For all $t \in \mathbb{Z}$ express $c_t = a_t * b_t$ as $(k_1\alpha^t + k_2\beta^t)u_t$ for suitable k_1, k_2.
   (b) Compute the DTFT : $A(\cdot)$, $B(\cdot)$, $C(\dot{})$ of a, b, c respectively.
   (c) Verify that $C(f) = A(f)B(f)$ for $f \in \mathbb{R}$.

Discrete Time Fourier Transform for infinite sequences- Filters

1. Show that
$$\sum_{t=-\infty}^{\infty} a * b_t e^{-i2\pi ft} = A(f)B(f)$$

2. (High Pass Filter) Suppose
$$a_t = \begin{cases} 1/2, & \text{t=0;} \\ -1/4, & t = \pm 1; \\ 0, & \text{otherwise,} \end{cases} \quad \text{and } b_t = \frac{3}{16}\left(\frac{4}{5}\right)^{|t|} + \frac{1}{20}\left(-\frac{4}{5}\right)^{|t|}, t \in \mathbb{Z}$$
   (a) Find $A(f)$ and $B(f)$
   (b) Plot $A(f)B(f)$.

3. (Low Pass Filter) Suppose
$$a_t = \begin{cases} 1/2, & \text{t=0;} \\ 1/4, & t = \pm 1; \\ 0, & \text{otherwise.} \end{cases} \quad \text{and } b_t = \frac{3}{16}\left(\frac{4}{5}\right)^{|t|} + \frac{1}{20}\left(-\frac{4}{5}\right)^{|t|}, t \in \mathbb{Z}$$
   (a) Find $A(f)$ and $B(f)$
   (b) Plot $A(f)B(f)$.

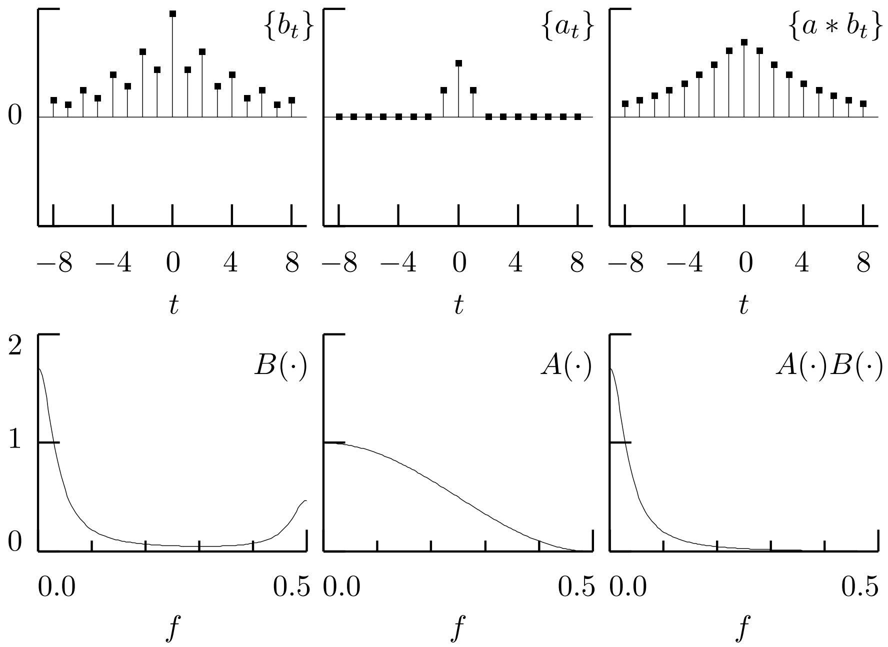


Figure 26. Example of filtering using a low-pass filter.

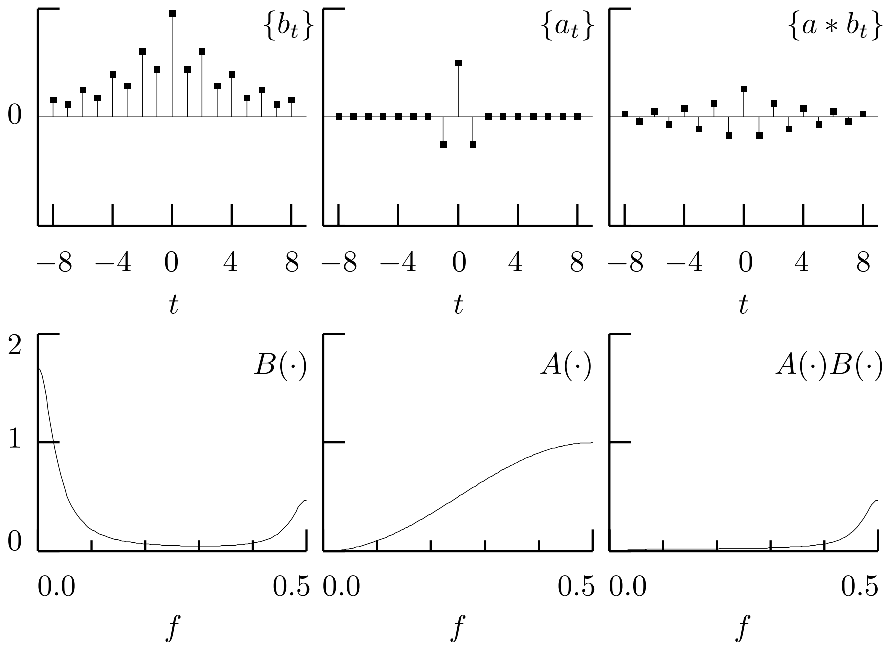
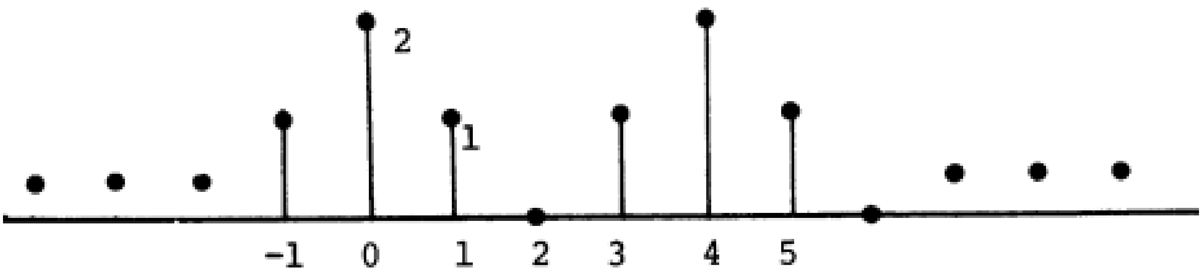


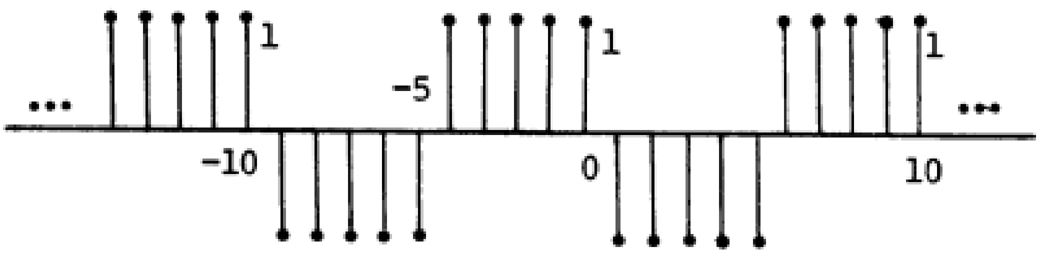
Figure 27. Example of filtering using a high-pass filter.

1. The below graph describes the period 3, sequence $\{\tilde{x}_n\}_{n\in\mathbb{Z}}$,



   Determine the Fourier Coefficients $\tilde{X}_k : k = 0, 1, 2, 3$.

2. The below graph describes the periodic sequence $\{\tilde{x}_n\}_{n\in\mathbb{Z}}$,



   Without explicitly evaluating the Fourier coefficients $\tilde{X}$. decide which of the following are true

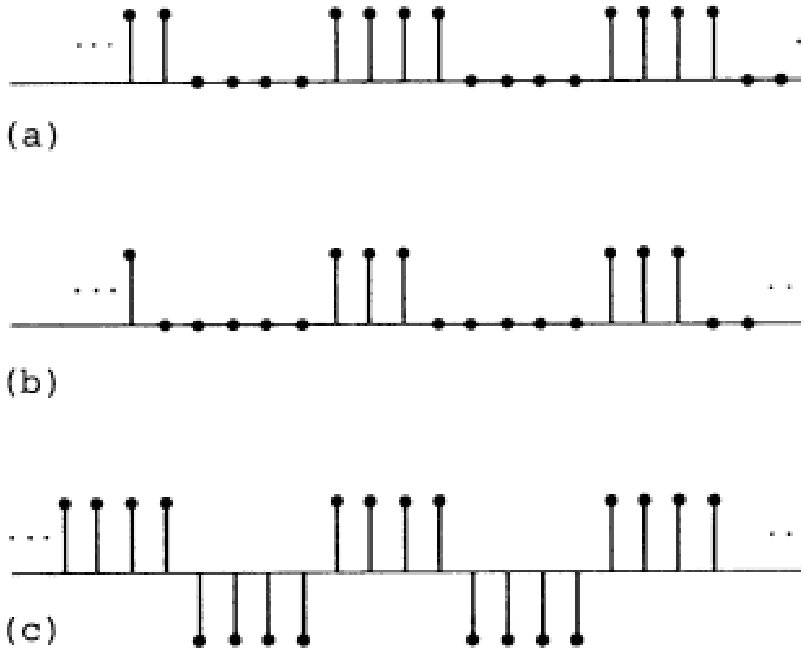
   (a) $\tilde{X}_k = \tilde{X}_{k+10}$ for all k.

   (b) $\tilde{X}_k = \tilde{X}_{-k}$ for all k.

   (c) $\tilde{X}_0 = 0$.

   (d) $\tilde{X}_k = \exp(i\frac{2\pi k}{5}$ is real for all k.

3. The below graph describes three periodic sequence $\{\tilde{x}_n\}_{n\in\mathbb{Z}}$,



Suppose the sequences can be expressed in terms of their Fourier coefficients as

$$\tilde{x}_n = \frac{1}{N}\sum_{k=0}^{N-1} X_k \exp(i\frac{2\pi nk}{N})$$

(a) For which sequences can the time origin be chosen such that all the X_k are real ?

(b) For which sequences can the time origin be chosen such that all the X_k for $k \neq 0$ are imaginary ?

1. *(Reverse Odd town)* Let $X = \{1, 2, ..., n\}$, and $A_1, A_2..., A_m$ be subsets of X with the property that for all i, $|A_i| =$ even, and $|A_i \cap A_j| =$ odd, for all $i \neq j$. Prove that $m \leq n$ (as usual $|A|$ denotes cardinality of the set A).

2. Let X be as in problem 1, and $A_1, A_2 ..., A_m$ be subsets of X with the property that $|A_i \cap A_j| = k$, for all $i \neq j$, where $1 \leq k < n$. Let v_i be the incidence vector of A_i. Show directly from the definition that $\{v_1, v_2 ..., v_m\}$ is a linearly independent set over $\mathbb{R}$, hence giving another proof that $m \leq n$.

3. Prove that in *any* coloring of the edges of K_6 by red or blue, one can always find a red K_3, or a blue K_3. Prove that K_6 cannot be replaced by K_5. You have just shown that $R(3,3) = 6$.

4.Demonstrate the inequality that $R(t,t) > (t-1)^2$ by explicilty constructing a two-coloring.

5. Fix a $z \in \mathbb{C}$. Consider $x_t = z^t$ for all $t \in \mathbb{Z}$.

   (a) Show that for any sequence $\{a_t\}_{t\in\mathbb{Z}}$, we have

   $$a_t * x_t = \sum_{s\in\mathbb{Z}} a_s x_{t-s} = c x_t$$

   for all $t \in \mathbb{Z}$ with c being a constant. What would you call x to be ?

   (b) If $\tilde{x}_t = z^t u_t$ for all $t \in \mathbb{Z}$ then does $\tilde{x}$ have the same property that x has ?