1. Consider the following two matrices in $GL_2(\mathbb{R})$:
$$a = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$
and
$$b = \begin{bmatrix} 0 & -1 \\ 1 & 0. \end{bmatrix}$$
Show that a has order 3 and b has order 4. What is the order of ab?

2. Give examples of the following:
(1) An infinite group G with a subgroup $H \neq G$ and $|G : H|$ finite.
(2) An infinite group G with a subgroup $H \neq \{e\}$ and $o(H)$ finite.

3. Show that the only finite group with 2 conjugacy classes is $\mathbb{Z}_2$.

4. Find the number of subgroups of the group $\mathbb{Z}/60\mathbb{Z}$.

5. Prove or disprove that if G is a finite abelian group of order n, and k is a positive integer which divides n, then G has atmost one subgroup of order k.

6. Find the number of elements of S_5 which are their own inverses.

7. Find the number of 2×2 matrices over $\mathbb{Z}_3$ with determinant 1.

8. For $a \in G$, let $\phi_a : G \longrightarrow G$ be the automorphism where $\phi_a(g) = aga^{-1}, \forall g \in G$. Let $Inn(G) = \{\phi_a | a \in G\}$. Show that
(i) $Inn(G)$ is a normal subgroup of $Aut(G)$.
(ii) $Aut(G)/Inn(G) \cong Z(G)$.

9. Prove that a group of order p^2 is abelian, where p is a prime.

10. Find the order of the element $\frac{2}{3} + \mathbb{Z}$ in $\mathbb{Q}/\mathbb{Z}$.

11. Let p be a prime number. Let G be the group of all 2×2 matrices over $\mathbb{Z}_p$, with determinant 1. Find the order of G.

12. Let G be a finite group and ϕ be an automorphism of G such that $\phi(x) = x$ if and only if $x = e$. Prove that every $g \in G$ can be represented as $g = x^{-1}\phi(x)$ for some $x \in G$. Moreover if $\phi(\phi(x)) = x$ for every $x \in G$, then show that G is abelian.

13. Let G be a group of order 24. Find the number of automorphisms of G onto G.

14. Let S_n denote the group of permutations on the set $\{1, 2, \ldots, n\}$ under the composition of functions. For $n > 2$, let H be the smallest subgroup of S_n containing $(1, 2)$ and the n-cycle $(1, 2, \ldots, n)$. Then show that $H = S_n$.

15. Which of the following conditions on the group G implies that G is abelian.
    (i) Order of G is p^3 for some prime p.
    (ii) Every proper subgroup of G is cyclic.
    (iii) Every subgroup of G is normal.
    (iv) The function $f : G \longrightarrow G$ defined by $f(x) = x^{-1}, \forall x \in G$ is a homomorphism.