Let $\{X_t : t = 0, 1, 2, \ldots, N-1\}$ be a set of real(or complex) numbers (sometimes called finite sequence). Define its orthonormal discrete Fourier transform (ODFT) to be the sequence

$$F_k = \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} X_t e^{-i\frac{2\pi tk}{N}},\ k = 0, \ldots, N-1. \tag{2}$$

1. Suppose $X_t \in \mathbb{R}$ for all $t = 0, 1, 2, \ldots, N-1$.
   (a) Show that F_0 is real-valued.
   (b) Show that $F_{N-k} = F_k^*$ for $1 \leq k < \frac{N}{2}$.
   (c) If N is even then $F_{\frac{N}{2}}$ is real-valued when N is even.

2. Let $\mathcal{F}$ be the $N \times N$ martix whose (k, t)th element be given by

$$\frac{1}{\sqrt{N}} \exp(-i\frac{2\pi tk}{N}),$$

   where $0 \leq k, t \leq N-1$. Let $\mathbf{X}$ be the column vector formed with entries of X and $\mathbf{F}$ be the the column vector formed with the entries of F.
   (a) Verify that $\mathbf{F} = \mathcal{F}\mathbf{X}$.
   (b) Verify that $\mathcal{F}$ is a unitary matrix. That is

$$\mathcal{F}^H\mathcal{F} = I_N.$$

   (c) Conclude that $\mathbf{X} = \mathcal{F}^H\mathbf{F}$

3. Let $N = 16$. Can you plot the real part of $\mathcal{F}^H_{k*}$ as a function of * ?