May 26, 2019
SWMS-2018-Follow up Worksheet 1

Due : May 30th, 11am 2019

Instructions: This is worksheet 1 in the follow up school on SWMS-2018. Solve as many questions as you can. Please work by yourself and write down complete solutions to each of the problems. **Do NOT copy someone else's solution and from the internet.** Begin each new problem on a new page and your NAME on every page. Please scan and email your solutions via email on May 30th, 2019, 11am.

Real Analysis

Let us introduce **Logical Notation:**
- $\forall$ to mean for all;
- $\exists$ to mean there exists;
- $\implies$ to mean implies; and
- $\iff$ to mean equivalent.

Here is an example of usage of notation:
Statement : For all $\epsilon > 0$ there is an N such that for all $n \geq N$, $a_n \in (a - \epsilon, a + \epsilon)$.
Statement in logical Notation: $\forall\, \epsilon > 0$, $\exists\, N$ such that $\forall n \geq N$, $a_n \in (a - \epsilon, a + \epsilon)$.

A sequence $\{a_n\}_{n\geq 1}$ is a function $a : \mathbb{N} \to \mathbb{R}$. We shall denote this function in short as, "a sequence $\{a_n\}_{n\geq 1}$."

1. We say $\lim_{n\to\infty} a_n = a$ if

   For every $\epsilon > 0$ there exists $N > 0$ such that $\mid a_n - a \mid < \epsilon$ whenever $n \geq N$.

   (a) Provide an example of sequence that converges to a.
   (b) Write a logical statement that is equivalent to saying $\lim_{n\to\infty} a_n \neq a$
   (c) Provide an example of sequence that does not converges to a.
   (d) Provide an example of sequence that does not converges to any real number.
   (e) Write a logical statement that is equivalent to saying that the sequence a_n does not converge to any real number.

2. A sequence $\{a_n\}$ is a bounded sequence if there is a $M > 0$ such that a_n is in the interval $(-M, M)$ for all $n \in \mathbb{N}$.

   (a) Write a logical statement that is equivalent to saying that the sequence a_n is bounded.
   (b) Provide an example of a bounded sequence: which converges and which does not converge.
   (c) Write a logical statement that is equivalent to saying that the sequence a_n is not bounded.
   (d) Write a logical statement that is equivalent to saying that the sequence a_n diverges to ∞.
   (e) Write a logical statement that is equivalent to saying that the sequence a_n diverges to $-\infty$.
   (f) Provide an example of a sequence : that is not bounded and diverges to $+\infty$; that is not bounded diverges to $-\infty$; and that is not bounded and neither diverges to $\pm\infty$.

(g) Provide an example (if any) of an ubounded sequence that converges to 0.

3. Find an example of a sequence that satisfies the below statements and then write the below statements using logical notation:

   (a) For every $\epsilon > 0$ there *are infinitely many* n such that distance of a_n to 0 is less than ϵ.

   (b) For every $\epsilon > 0$ *for all but finitely many* n the distance of a_n to 0 is less than ϵ.

   (c) For every $\epsilon > 0$, all but finitely many elements of the sequence a_n are below $11+\epsilon$ and inifinitely many above $11 - \epsilon$.

4. Provide two examples (if any) of sequences $\{a_n\}$ that satisfy each of the statements below.

   (a) A sequence that converges to 0 which has the property that infinitely many elements are negative numbers and infinitely many elements are positive numbers.

   (b) A non-constant sequence that does not converge.

   (c) For all $\epsilon > 0$, there are infinitely many $n \in \mathbb{N}$ such that

$$| a_n - L | < \epsilon$$

   for $L = -1, 0, 3$ and $a_n \notin \{-1, 0, 3\}$ for infinitely many $n \in \mathbb{N}$.

   (d) A sequence that converges to 0 which has the property that all but finitely many elements are negative.

   (e) For all $\epsilon > 0$, for all but finitely many $n \in \mathbb{N}$ $a_n < 5 + \epsilon$ and $a_n > -11 - \epsilon$.

   (f) For all $M > 0$:
   there are infinitely many $n \in \mathbb{N}$ such that $a_n > M$ and
   there are infinitely many $n \in \mathbb{N}$ such that $a_n < -M$.

Negation of statement A: a statement B whose assertion specifically denies the truth of statement A.

5. Negate the below statements and express the negations in English avoiding the use of negation words whenever possible.

   (a) All classrooms in the ICTS main building have at least one chair that is broken.

   (b) No classroom in the ground floor has only chairs that are not broken.

   (c) Every student in this class has taken Mathematics or Physics in Class XII.

   (d) Every student in this class has taken Mathematics and Biology in Class XII.

   (e) In every batch of SWMS there is a student who has taken neither Mathematics nor Biology in high school.

6. Let us define the following terms:-

   - $S \subseteq R$ is bounded if there exists M such that $| x | \leq M$ for all $x \in S$.
   - $f : \mathbb{R} \to \mathbb{R}$ is bounded if there exists M such that $| f(x) | \leq M$ for all $x \in \mathbb{R}$.
   - f is increasing (or strictly increasing) if $f(x) < f(y)$ whenever $x < y$.
   - f is nondecreasing (or weakly increasing) if $f(x) \leq f(y)$ whenever $x < y$.
   - f is decreasing (or strictly decreasing) if $f(x) > f(y)$ whenever $x < y$.
   - f is nonincreasing (or weakly decreasing) if $f(x) \geq f(y)$ whenever $x < y$.

   (a) Provide an example of a set $S \subseteq \mathbb{R}$ that is bounded and $f : \mathbb{R} \to \mathbb{R}$ that is bounded.

   (b) Write a logical statement that is equivalent to saying $f : \mathbb{R} \to \mathbb{R}$ is not bounded.

   (c) Write a logical statement that is equivalent to saying $S \subseteq \mathbb{R}$ is not bounded.

   (d) Provide an example of a set $S \subseteq \mathbb{R}$ that is not bounded and $f : \mathbb{R} \to \mathbb{R}$ that is not bounded.

(e) If $f : \mathbb{R} \to \mathbb{R}$ is unbounded then construct a sequence $\{a_n\}$ such that $\lim_{n\to\infty} | f(a_n) |= \infty$

7. Let A, B, C be non-empty sets. Let $f : A \to B$, $g : B \to C$ and $h : A \to C$ given by $h = g \circ f$. Decide which of the following statements are true or false.

   (a) If f and g are injective, then h is injective.

   (b) If f and g are surjective, then h is surjective.

   (c) If h is injective, then f is injective.

   (d) If h is injective, then g is injective.

   (e) If h is surjective, then f is surjective.

   (f) If h is surjective, then g is surjective.

Probability

8. Suppose that n people, of which k are men, are arranged at random in a line. What is the probability that all the men end up standing next to each other?

9. An assembler of electric fans uses motors from two sources. Company A supplies 90% of the motors, and company B supplies the other 10%. Suppose that 5% of the motors supplied by company A are defective and that 3% of the motors supplied by company B are defective. An assembled fan is found to have a defective motor. What is the probability that this motor was supplied by company B ?

10. Solve the following questions and giving **reasons** for your answer.

    (a) Let X be a discrete random variable. Which of the following functions can represent the distribution function F of X:-

(i)	**(ii)**	**(iii)**
$F(x) = \begin{cases} 0 & x \leq -1 \\ 0.6 & -1 < x < 1 \\ 1 & x \geq 1 \end{cases}$	$F(x) = \begin{cases} 0 & x < 0 \\ 0.5 & 0 \leq x < 1 \\ 1 & x \geq 1 \end{cases}$	$F(x) = \begin{cases} 0 & x < 0 \\ 0.4 & 0 \leq x < 1 \\ 0.3 & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$

    (b) Decide whether the following statement is true:- "If A, B, and C are pairwise independent events then they are independent events."

    (d) Let $X \stackrel{d}{=}$ Binomial $(36, \frac{1}{2})$. Let $\Phi(t) = \int_{-\infty}^{t} dt \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}}$. Find the r, s, t, u, v so that the following approximations are valid:
    (i) $P(3 \leq X \leq 20) \approx \Phi(r) - \Phi(s)$
    (ii) $P(X = 20) \approx \dfrac{e^{-t} u^{20}}{v}$

11. Let X be a Geometric $(\frac{1}{2})$ random variable and Y be an independent Geometric$(\frac{1}{2})$ random variable. Let $W = \max\{X, Y\}$. Find the distribution of W.

12. A box contains 10 coins, of which 5 are fair and 5 are biased to land heads with probability 0.7. A coin is drawn from the box and tossed once.

    (a) What is the chance that it will land a head?

    (b) Suppose that the coin drawn landed a head. Given this information, what is the conditional probability that if I draw another coin from the box(without replacing the first coin), then that coin will be a fair coin ?

13. Let $X_1, \dots X_n, \dots$ be a sequence of independent and identically distributed random variables, with $E(X_i) = 10$ and $V(X_i) = 1$.

    (a) State the weak law of large numbers for the above mentioned sequence $X_1, \dots X_n, \dots$.

    (b) Let $T_n = \frac{10}{n} \sum_{i=1}^{n} (30 + X_i)$. Does T_n converge to something and if so what is type of convergence ?

14. Let X be a $\Gamma(4, 2)$ random variable with moment generating function $M_X(t)$. Let Y be another random variable with moment generating function $M_Y(t)$. Suppose $M_X(t) = M_Y(4t)$, then

    (a) Find the relationship between X and Y.

    (b) Find the probability density function of Y.

15. Waiting times at a service counter in a pharmacy are exponentially distributed with a mean of 10 minutes.

    (a) What is the probability that one customer has to wait for more than 10 minutes ?

    (b) Let 100 customers come to the service counter in a day. Let S_{100} be the number of customers that wait for more than 10 minutes.

        (i) Write out the probability mass function of S_{100}.

        (ii) Using the central limit theorem, approximate the probability that at least half of the customers that arrive in a day must wait for more than 10 minutes. Please explain clearly how the central limit theorem is used in the solution.

16. Let X and Y be independent random variables each geometrically distributed with parameter p.

    (a) Find $P(\min(X, Y) = X)$.

    (b) Find the distribution of $X + Y$.

    (c) Find $P(Y = y | X + Y = z)$.

Algebra

17. Let G_1 be a group of order 21 and G_2 be a group of order 49. Suppose G_1 does not contain a normal subgroup of order 3. Find all homomorphisms between G_1 and G_2.

18. Let $(\mathbb{Q}, +)$ be the additive group of rational numbers and let $\mathbb{Q}^+, \times)$ be the multiplicative group of positive rational numbers. Prove that they are not isomorphic as groups.

19. Prove that every finite group having more than two elements has a nontrivial automorphism.

20. Let G be a group of order p^n, where p is a prime, $n \geq 1$. Show that a subgroup of index p in G is normal.

Vector Spaces

21. Let $\mathcal{P}_2(\mathbb{R})$ be the vector space over $\mathbb{R}$ consisting of all polynomials with real coefficients of degree 2 or less. Let $B = \{1, x, x^2\}$ be a basis of the vector space $\mathcal{P}_2(\mathbb{R})$. For each linear transformation $T : \mathcal{P}_2(\mathbb{R}) \longrightarrow \mathcal{P}_2(\mathbb{R})$ defined below, find the matrix representation of T with respect to the basis B. For $f(x) \in \mathcal{P}_2(\mathbb{R})$, define T as follows.
    (a) $T(f(x)) = \frac{d^2}{dx^2} f(x) - 3 \frac{d}{dx} f(x)$,
    (b) $T(f(x)) = \int_{-1}^{1} (t - x)^2 f(t) dt$.

22. Let V be the vector space of all 3×3 real matrices. Let

$$A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix},$$

where a, b, c are distinct real numbers. Define

$$W = \{B \in V | AB = BA\}.$$

(i) Show that W is a subspace of V.
(ii) Find the dimension of W.

23. Show that $\{a, b, c\}$ is a basis of $\mathbb{R}^3$ where

$$a = (-1, 1, 1), b = (1, -1, 1), c = (1, 1, -1).$$

Let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ be the linear transformation such that
$f(a) = (1, 0, 1, \lambda)$, $f(b) = (0, 1, -1, 0)$, $f(c) = (1, -1, \lambda, -1)$.
(i) Determine $f(x, y, z)$ for all $(x, y, z) \in \mathbb{R}^3$.
(ii) For which values of λ is f injective?
(iii) Consider the subspace W of $\mathbb{R}^4$ given by $W = \text{Span } \{f(a), f(b)\}$. Determine $\dim(W)$ when $\lambda = -1$.

24. For the matrix

$$A = \begin{bmatrix} 2 & 0 & 4 \\ 3 & -4 & 12 \\ 1 & -2 & 5 \end{bmatrix}$$

determine an invertible matrix P such that $P^{-1}AP$ is diagonal.