Guardian's address:

Current status:

- With regard to goals, please mention your aims within the program that you are in. For example, someone in BSc mathematics could say: I aim at 85% average or top 2% of the class.

- If you are in a program and you need to choose your major subject to get your degree in then please mention topics in mathematics that you would like covered in the summer school that will help you towards your major.

- If you are about to enter the final year of your program then please mention the degree programs (if any) that you will be applying for.

February 15, 2019
SWMS-2018-Follow up School Selection Test

Due : March 31st, 2019

Instructions: These questions are on pre-requisite material to follow up school on SWMS-2018[1] and selection will be based on performance in these. Solve as many questions as you can. Please work by yourself and write down complete solutions to each of the problems. **Do NOT copy someone else's solution.** Begin each new problem on a new page and your NAME on every page.

1. (**Calculus**) We say $f : [0,1] \to \mathbb{R}$ is continuous at $c = \frac{1}{2}$ if

   for every $\epsilon > 0$ there exists $\delta > 0$ such that $\mid f(x) - f(c) \mid < \epsilon$ whenever $x \in [0,1]$ and $\mid x - c \mid < \delta$.

   Consider the following statements:

   (a) For every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in [0,1]$, $\mid x - c \mid < \delta$ implies $\mid f(x) - f(c) \mid < \epsilon$.

   (b) For every $\delta > 0$ there exists $\epsilon > 0$ such that for all $x \in [0,1]$, $\mid x - c \mid < \delta$ implies $\mid f(x) - f(c) \mid < \epsilon$.

   (c) There exists $\delta > 0$ such that for all $\epsilon > 0$ and for all $x \in [0,1]$, $\mid x - c \mid < \delta$ implies $\mid f(x) - f(c) \mid < \epsilon$.

   (d) For every $\epsilon > 0$ and for all $x \in [0,1]$, there exists $\delta > 0$ such that $\mid x - c \mid < \delta$ implies $\mid f(x) - f(c) \mid < \epsilon$.

   Decide which of the above versions are equivalent to the definition of $f : [0,1] \to \mathbb{R}$ is continuous at $c = 0.5$. If verstion is equivalent then provide a proof. If it is not then describe in as simple a language as possible, what they really require and provide an example (if they exist) of function that satisfy them.

2. (**Calculus**) For any two non-empty sets A, B, We say that Card (A) = Card (B) if there is a bijection $f : A \to B$. Decide if Card (A) = Card (B) in the cases below.

   (a) $A = (0,1)$ and $B = [0,1]$

   (b) $A = \mathbb{N}$ and $B = \mathbb{R}$.

   Show that it is not possible to have a $f : (0,1) \to [0,1]$ that is a continuous bijection.

3. (**Probability**) Each morning at 7 a.m., Natasha Bakery brings out fresh chocolate cup cakes on its shelf, filled with preservatives to last a lifetime. Suppose that the number of cup cakes brought out each morning has a Poisson distribution with parameter 10 and is independent of what has gone on in the past. The probability of any given cup cake is sold in any given 24 hour period is p (independent of day and other cup cakes). Let X_i be the number of unsold cupcakes on i^{th} morning at Natasha Bakery before Natasha brings out her new batch. Assume $X_1 = 0$.

   (a) What is the distribution of X_2 ?

   (b) What is the distribution of X_n for $n > 3$?

   (c) What will be the expected number of unsold cup cakes by day 100 ?

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[1] to be held in May-June 2019

4. (**Probability**) Continuous random variables X and Y have a joint density

$$f(x,y) = \begin{cases} k, & \text{for } 0 < x < 6, 0 < y < 4, \\ 0, & \text{otherwise.} \end{cases}$$

   (a) Find k.
   (b) Find $P(2Y > X)$.
   (c) Find the marginal density of X.
   (d) Find the conditional density of Y given $X = 2$.
   (e) Are X and Y independent?
   (f) Find $E((X+Y)^2)$.

5. (**Statistics**) In Siva's Statistics course, the correlation between the students total scores before the final examination and their final examination score is $r = .6$. The total before the exams for all students in the course have mean 280 and standard deviation 30. The final exam scores have mean 75 and standard deviation 8. Siva has lost Soha's final exam but knows that her total before the exam was 300. He decided to predict her final-exam score from her pre-exam total.
   (a) What is the slope of the least square regression line of final-exam scores on pre-exam scores in this course? What is the intercept?
   (b) Use the regression line to predict Julies Final exam score.
   (c) Soha complains to the Student's Dean as she doesnt think this method accurately predicts how well she did on the final exam. Using R^2 statistic decide if the Student's Dean should consider her complaint.

6. (**Statistics**) You want to see if listening to Kishore Kumar songs while you study is helpful, detrimental, or neutral for SWMS students.
   (a) Briefly explain how you would collect data to answer this question.
   (b) A student collected data on this in the last summer school, and when testing these hypotheses, she got a p-value of 0.08. Using $\alpha = 0.05$, state the conclusion in context.
   (c) If listening to Kishore Kumar songs does not affect studying, did she make an error? If so,what type?
   (d) If listening to Kishore Kumar songs does affect studying, did she make an error? If so, whattype?
   (e) How could the student have decreased her chance of making a Type I error?(f)(2)How could the student have decreased her chance of making a Type II error?

7. (**Linear Algebra**) Think of $\mathbb{R}^{n\times n}$ as the space of all $n \times n$ matrices with real entries. The **trace** of a matrix is the sum of its diagonal entries. Let W_1 be the space of $n \times n$ matrices with trace zero. Find a subspace W_2 of $\mathbb{R}^{n\times n}$ such that $\mathbb{R}^{n\times n} = W_1 \oplus W_2$.

8. (**Linear Algebra**)
   (a) Let A, B be matrices of size $m \times r$ and $r \times n$ respectively. Prove that the rank of AB is at most r.
   (b) Let A be an $m \times n$ matrix of rank r. Prove that there exist matrices X, Y of size $m \times r$ and $r \times n$ respectively such that $A = XY$.

9. (**Linear Algebra**) Given any two nonzero vectors $v, w \in \mathbb{R}^n$, prove that there exists an invertible $n \times n$ real invertible matrix A such that $Av = w$. Does the result hold if $\mathbb{R}$ is replaced by any field F?

10. (**Linear Algebra**) Give examples of 2 square matrices A, B of the same size such that their characteristic as well as minimal polynomials are equal, but A is not similar to B. Justify your answer.

11. (**Abstract Algebra**) Let $\mathbb{C}^\times$ denote the multiplicative group of non-zero complex numbers. Show that the quotient group $\mathbb{R}/\mathbb{Z}$ is isomorphic to the circle group $S^1 := \{z \in \mathbb{C} : |z| = 1\} \subset \mathbb{C}^\times$. What is the image of the subgroup $\mathbb{Q}/\mathbb{Z}$, under this isomorphism? (Hint: Use first isomorphism theorem)

12. (**Abstract Algebra**) Prove that an abelian group G of order pq, where p and q are distinct primes, is cyclic.