1. Let $A \subset \mathbb{R}$. Define:
   (a) What is meant by saying A is a bounded subset of $\mathbb{R}$?
   (b) Negate the above statement using logical notation.
   (c) What is meant by saying $\alpha = \sup(A)$ and $\beta = \inf(A)$?

   Solution: See Class Board Photos.[Page 1-6]

2. Find the infimum and supremum of the sets
   (a) $B = \{2, 3, 4\}$
   (b) $S = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{50 + \frac{1}{n} : n \in \mathbb{N}\}$
   (c) $A = \{1 - \frac{(-1)^n}{n} : n \in \mathbb{N}\}$

   Solution: See Class Board Photos.[Page 7-10]

3. Let $A = \{p \in \mathbb{Q} : p^2 < 2\}$. Show that A is bounded above but does not have a least upper bound in $\mathbb{Q}$

   Solution: See Principles of Mathematical Analysis by W. Rudin, Page 11,12

4. Let A and B be two subsets of $\mathbb{R}$, which are both bounded below. Let $u = \inf(A)$ and $v = \inf(B)$. Find $\inf(C)$, in terms of u, v, when
   (a) $C = \{a + b : a \in A, b \in B\}$,
   (b) $C = A \cup B$,
   (c) $C = A \cup \{10\}$.

   Solution: For 4(a) See Class Board Photos.[Page 11-12] For 4(b) Ans: $\min\{u, v\}$ and 4(c) Ans: $\min\{u, 10\}$

5. Let $n \geq 1$ and $x_i > 0$. Prove the Bernoulli inequality:

$$\prod_{i=1}^{n}(1 + x_i) \geq 1 + \sum_{i=1}^{n} x_i$$

6. If $a \in \mathbb{R}$ such that $0 \leq a < \epsilon$ for every $\epsilon > 0$, then show that $a = 0$.

   Solution: We will prove this by contradiction. Suppose $a \neq 0$. Then, since it is given that $a \geq 0$, $a > 0$. By taking $\epsilon = a$, we conclude that $a < a$, which is a contradiction. Hence $a = 0$.‘

   $\square$

1. Let A and B be two subsets of $\mathbb{R}$, which are both bounded below. Let $u = \sup(A)$ and $v = \sup(B)$. Find $\sup(C)$, in terms of u, v, when
   (a) $C = \{a + b : a \in A, b \in B\}$,
   (b) $C = A \cup B$, (What if $C = A \cap B$?)
   (c) $C = A \cup \{-10\}$.

   Solution: See Related Problem for 1(a)

2. Let A and B be two non-empty subsets of $\mathbb{R}$. Define what is meant by a function $f : A \to B$ and when is it called one-one and onto. Provide examples of f that are (and are not) one-one and (or) onto when A, B are finite or countable or uncountable.

   Solution: : Check E.B. Folland, Real Analysis

1. In each of the cases below decide if $\{x_n\}_{n=1}^{\infty}$converges or not:
   (a) $x_n = \frac{1}{n}$
   (b) $x_n = \sqrt{n^2 - n} - n$
   (c) $x_n = \frac{2^n}{n!}$,
   (d) $x_n = nb^n$, for $b \in (0, 1)$

   Solution: See Class Board Photos.[Page 14-18]

2. Let a_n be a bounded sequence of real numbers. Let $s = \sup\{a_n : n \geq 1\}$ and $s \notin \{a_n : n \in \mathbb{N}\}$. Show that there is a subsequence of a_n that increases to s.

   Solution: Note that
   $$s = \sup\{a_n : n \geq 1\} \text{ and } s \notin \{a_n : n \in \mathbb{N}\}. \tag{1}$$
   From (1) it is immediate that there exists $n_1 > 0$ such that
   $$s - 1 < a_{n_1} < s.$$
   Let $\epsilon_2 = \frac{1}{2}\min\{1, \min\{| s - a_i |: 1 \leq i \leq n_1\}$. Using (1) again we have there exists $n_2 > 0$ such that
   $$s - \epsilon_2 < a_{n_2} < s.$$
   By definition of ϵ_2 we have that
   $$n_2 > n_1, a_{n_2} > a_{n_1}, \text{ and } s - \frac{1}{2} < a_{n_2} < s.$$
   Thus inductively (why ?) we can construct for each $k \geq 1$
   $$n_k > n_{k-1}, a_{n_k} > a_{n_{k-1}}, \text{ and } s - \frac{1}{k} < a_{n_k} < s.$$
   By construction a_{n_k} is a subsequence of a_n, it is increasing, and $a_{n_k} \to s$ as $k \to \infty$. □

3. Suppose $\{a_n\}_{n \geq 1}$ is a sequence of real numbers such that
   $$\lim_{n \to \infty} | a_{n+1} - a_n |= 0.$$
   Does this necessarily imply that the sequence is a Cauchy sequence.

   Solution: Answer is No. See Class Board Photos, Page 2 (upside down)

4. Consider the $\{y_n\}_{n=1}^{\infty}$, such that $y_1 > 1$ and $y_{n+1} := 2 - \frac{1}{y_n}$ for $n \geq 2$. Show that y_n converges.

5. (*Finding Roots of a number*) Let $a > 0$ and choose $s_1 > \sqrt{a}$. Define
   $$s_{n+1} := \frac{1}{2}(s_n + \frac{a}{s_n})$$
   for $n \in \mathbb{N}$.
   (a) Show that s_n is monotonically decreasing and $\lim_{n \to} s_n = \sqrt{a}$.
   (b) If $z_n = s_n - \sqrt{a}$ then show that $z_{n+1} < \frac{z_n^2}{2\sqrt{a}}$.
   (c) Let $f(x) = x^2 - a$. Show that $s_n = s_{n-1} - \frac{f(s_{n-1})}{f'(s_{n-1})}$.
   (d) Draw graph of f with $a = 4$ and plot the sequence s_n for a few steps when $s_0 = 5$.

   Solution: See Class Board Photos.

1. Decide if $\{x_n\}_{n=1}^{\infty}$either converges to a real number or diverges to ∞ or diverges to $-\infty$ or none of the above when $x_n = n^{\frac{1}{n^2}}$
2. Let $x \in \mathbb{R}$, $\{x_n\}_{n=1}^{\infty}$be a sequence of real numbers. Show that the following are equivalent:
   (a) $\forall \epsilon > 0$, there is an $N \in \mathbb{N}$ such that $\mid x_n - x \mid < \epsilon$ for all $n \geq N$.
   (b) $\forall \epsilon \in (0, 1)$, there is an $N \in \mathbb{N}$ such that $\mid x_n - x \mid < \epsilon$ for all $n \geq N$.
   (c) Let $C > 0$, $\forall \epsilon > 0$, there is an $N \in \mathbb{N}$ such that $\mid x_n - x \mid \leq C\epsilon$ for all $n > N$.

1. Let A be a non-empty set of real numbers which is bounded below. Let $-A := \{x \in \mathbb{R} : -x \in A\}$. Show that $\inf(A) = -\sup(-A)$.

2. If $z, w, z_i \in \mathbb{C}$ for $i = 1, 2, \ldots, n$ then show that

$$|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$$

and

$$||z| - |w|| \leq |z - w|$$

Solution: We will prove the first inequality by induction on n. Let $\mathcal{P}(n)$ be the statement:

$$\left|\sum_{i=1}^{n} z_i\right| \leq \sum_{i=1}^{n} |z_i|$$

$\mathcal{P}(1)$ is true as $z_1 \leq z_1$. We have seen that $\mathcal{P}(2)$ is true. Let $\mathcal{P}(k)$ be true for some $k \in \mathbb{N}$ such that $k \geq 2$. We will show that $\mathcal{P}(k+1)$ is true.

$$\left|\sum_{i=1}^{k+1} z_i\right| = \left|\left(\sum_{i=1}^{k} z_i\right) + z_{k+1}\right| \leq \left|\sum_{i=1}^{k} z_i\right| + |z_{k+1}| \leq \left(\sum_{i=1}^{k} |z_i|\right) + |z_{k+1}| = \sum_{i=1}^{k+1} |z_i|$$

This proves $\mathcal{P}(k+1)$. We have used both $\mathcal{P}(2)$ and $\mathcal{P}(k)$ above. $\mathcal{P}(1)$ is true, $\mathcal{P}(2)$ is true, and $\mathcal{P}(k) \implies \mathcal{P}(k+1)\ \forall\ k \in \mathbb{N} - \{1\}$. Hence by the principle of mathematical induction, $\mathcal{P}(n)$ is true for all $n \in \mathbb{N}$.

Now we will prove the second inequality. For any $a, b \in \mathbb{R}$,

$$|a + ib| = \sqrt{a^2 + b^2} = \sqrt{(-a)^2 + (-b)^2} = |-(a + ib)|$$

Hence $|z_1| = |-z_1|\ \forall\ z \in \mathbb{C}$. In particular, $|z - w| = |w - z|$. Now,

$$|z| = |(z - w) + w| \leq |z - w| + |w|$$

$$|z| - |w| \leq |z - w|$$

Interchanging w and z,

$$|w| - |z| \leq |w - z| = |z - w|$$

Since $|w|$ and $|z|$ are reals, $||z| - |w||$ equals at least one of $|z| - |w|$ and $|w| - |z|$, both of which are less than or equal to $|z - w|$. Hence,

$$||z| - |w|| \leq |z - w|$$

□

3. Let A and B be bounded nonempty subsets of $\mathbb{R}$, and let $A + B := \{a + b : a \in A, b \in B\}$. Prove that $\sup(A + B) = \sup(A) + \sup(B)$.

Solution: We will first prove that $\sup(A+B) = \sup(A)+\sup(B)$. Let $\alpha = \sup(A)$ and $\beta = \sup(B)$. As $\alpha \geq a\ \forall\ a \in A$ and $\beta \geq b\ \forall\ b \in B$, we conclude that $\alpha + \beta \geq a + b\ \forall\ a \in A, b \in B$. Hence $\alpha + \beta \geq x\ \forall\ x \in (A + B)$. $\alpha + \beta$ is an upper bound of $A + B$.

Let $\gamma < \alpha + \beta$. To prove that $\alpha + \beta$ is the supremum of $A + B$, we need show that γ cannot be an upper bound of $A + B$, i.e. we need to show that there exists $x \in A + B$ such that $x > \gamma$. Let $p = (\alpha + \beta - \gamma)/2$. Note that $p > 0$. As $\alpha - p$ is not an upper bound of A, $\exists\ a' \in A \ni a' > \alpha - p$.

As $\beta - p$ is not an upper bound of B, $\exists\ b' \in B \ni b' > \beta - p$. Now, $a' + b' \in A + B$. Also, $a' + b' > (\alpha - p) + (\beta - p) = \gamma$. So we have found $x = a' + b' \in A + B$ such that $x > \gamma$. Hence $\alpha + \beta$ is the supremum of $A + B$. $\sup(A + B) = \sup(A) + \sup(B)$.

For any $X \subseteq \mathbb{R}$, define $-X := \{-x : x \in X\}$. So, for any $y \in \mathbb{R}$, $y \in X \iff (-y) \in (-X)$.

$$\begin{aligned} (-A) + (-B) &= \{x + y : x \in (-A), y \in (-B)\} \\ &= \{(-x) + (-y) : x \in A, y \in B\} \\ &= \{-(x + y) : x \in A, y \in B\} \\ &= \{-z : z \in (A + B)\} \\ &= -(A + B) \end{aligned}$$

We know that if $X \subseteq \mathbb{R}$ is non-empty and bounded below, $\inf(X) = -\sup(-X)$ by below **Claim**. 1Hence, we also have

$$\begin{aligned} \inf(A + B) &= -\sup(-(A + B)) \\ &= -\sup((-A) + (-B)) \\ &= -(\sup(-A) + \sup(-B)) \\ &= -\sup(-A) - \sup(-B) \\ &= \inf(A) + \inf(B) \end{aligned}$$

This completes the proof. $\square$

Claim: Let A be a non-empty set of real numbers which is bounded below. Let $-A := \{x \in \mathbb{R} : -x \in A\}$. Show that $\inf(A) = -\sup(-A)$.

Solution: : A is non-empty and bounded below, so it has a greatest lower bound. Let $\alpha = \inf(A)$. As α is a lower bound of A, $\alpha < x\ \forall\ x \in A$. Therefore $-\alpha > -x\ \forall\ x \in A$. As $x \in A \iff -x \in -A$, we conclude that $-\alpha > x\ \forall\ x \in -A$. This shows that $-\alpha$ is an upper bound of $-A$.

As α is the greatest lower bound of A, $\beta > \alpha \Rightarrow \exists\ x \in A \ni x < \beta$. Let $\gamma < -\alpha$. To show that $-\alpha$ is the least upper bound of $-A$, we need to show that $\exists\ y \in -A \ni y > \gamma$. By taking $\beta = -\gamma$, we see that $\exists\ x \in A \ni x < -\gamma$. $-x \in -A$ and $-x > \gamma$. So we have found a y, namely $-x$. So we conclude that $-\alpha$ is the least upper bound of $-A$. $\inf(A) = -\sup(-A)$.

$\square$

4. In each of the cases below decide if $\{x_n\}_{n=1}^{\infty}$either converges to a real number or diverges to ∞ or diverges to $-\infty$ or none of the above.

   (a) $x_n = \frac{2n^2+1}{3n^2-1}$
   (b) $x_n = n^{(-1)^n}$
   (c) $x_n = \frac{n!}{n^n}$

1. Decide if $\{x_n\}_{n=1}^{\infty}$either converges to a real number or diverges to ∞ or diverges to $-\infty$ or none of the above when $x_n = (a_1^n + a_2^n + a_3^n)^{\frac{1}{n}}$, with $a_1, a_2, a_3 > 0$

2. Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence. Then show that it has a convergent subsequence.

3. Let $\{x_n\}_{n=1}^{\infty}$be a sequence of real numbers $\mathbb{R}$ and suppose that $x_n \to x$.

   (a) Let $m, n \in \mathbb{N}$, show that $x_{m+n} \to x$ as $m \to \infty$.

   (b) Let $m, l \in \mathbb{N}$, $p : \mathbb{R} \to \mathbb{R}$ such that $p(x) = \sum_{k=0}^{l} p_k x^k$, and $q : \mathbb{R} \to \mathbb{R}\backslash\{0\}$, $q(x) = \sum_{k=0}^{m} q_k x^k$, with $p_k \in \mathbb{R}, q_k \in \mathbb{R}$ for $k = 1, 2, \ldots, n$. Show that if $r : \mathbb{R} \to \mathbb{R}$ defined by $r(x) = \frac{p(x)}{q(x)}$ then $r(x_n) \to r(x)$.

   (c) Show that $\{\mid x_n \mid\}_{n=1}^{\infty}$ also converges

1. Let A and B be two subsets of $\mathbb{R}$, which are both bounded below. Let $u = \inf(A)$ and $v = \inf(B)$. Find $\inf(C)$, in terms of u, v, when $C = \{ab : a \in A, b \in B\}$ and $A, B \subset [0, \infty)$,

   Solution: See Night Board discussion[Page 1]

2. Let $\{x_n\}_{n=1}^{\infty}$be a sequence of real numbers that is not bounded. Then show that there is either a subsequence that diverges to ∞ or a subsequence that diverges to $-\infty$.

   Solution: See Night Board discussion[Page 2]

3. Decide if $\{x_n\}_{n=1}^{\infty}$converges or not when $x_n = \frac{n^\alpha}{(1+p)^n}$ with $\alpha \in \mathbb{R}, p > 0$.

   Solution: See Night Board discussion[Page 3]

4. Let $y_1, y_2 \in \mathbb{R}$ be given, and define recursively for $n \geq 1$,

$$y_{n+2} = \frac{1}{3}y_n + \frac{2}{3}y_{n+1}$$

   for all $n \in \mathbb{N}$. Decide if y_n converges and if it does then find its limiting value.