



$$1(b) S_n = \sum_{k=1}^n \frac{1}{k^p} \quad p < 1$$

$$(1) \quad \frac{1}{k^p} > \frac{1}{k} \quad \forall k \geq 1 \text{ \& } p < 1$$

$$(2) \Rightarrow \sum_{k=1}^n \frac{1}{k^p} > \sum_{k=1}^n \frac{1}{k} \quad \forall n \geq 1$$

In moving we showed
 $a_n = \sum_{k=1}^n \frac{1}{k}$ is not Cauchy sequence

Also know $a_n < a_{n+1} \quad \forall n \geq 1$
So $a_n \rightarrow \infty$ as $n \rightarrow \infty$ — (3)

$$\text{From (2)} \quad S_n > a_n \quad \text{--- (4)}$$

$$\text{From (3)} \quad \forall M > 0 \quad \exists N > 0 \quad a_n > M \quad \forall n \geq N$$

$$(4) \quad \forall M > 0 \quad \exists N > 0 \quad S_n > M \quad \forall n \geq N$$

($S_n > a_n$)

$$\Rightarrow S_n \rightarrow \infty \text{ as } n \rightarrow \infty$$

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$$0 \leq a_n \leq b_n \quad \forall n \geq 1 \quad \text{--- (1)}$$

$$\left. \begin{aligned} S_n &= \sum_{k=1}^n a_k, & S_{n+1} &= S_n + a_{n+1} \\ & & &\geq S_n \\ T_n &= \sum_{k=1}^n b_k, & T_{n+1} &= T_n + b_{n+1} \\ & & &\geq T_n \end{aligned} \right\} \text{--- (2)}$$

S_n and T_n are increasing sequences

Comparison Test

$$S_n \rightarrow \infty \text{ as } n \rightarrow \infty$$

$$\Rightarrow T_n \rightarrow \infty \text{ as } n \rightarrow \infty$$

Suppose $T_n \rightarrow t$ as $n \rightarrow \infty$
then there is an $s \in \mathbb{R}$
such that $S_n \rightarrow s$ as $n \rightarrow \infty$

$$\text{--- (3)}$$

From (1)

$$S_n \leq T_n \quad \forall n \geq 1$$

From (2) we have

$$S_n \leq t \quad \text{--- (3)}$$

From (1) & (3)

"there is an $s \in \mathbb{R}$ such that $S_n \rightarrow s$ as $n \rightarrow \infty$ "

$$S = \sum_{k=1}^{\infty} a_k$$



$\{a_n\}_{n \geq 1}$ be a sequence.

let $\{n_k\}_{k \geq 1}$ be a strictly increasing

sequence i.e. $n_k < n_{k+1} \quad \forall k \geq 1$

$\{a_{n_k}\}_{k \geq 1}$ is called a subsequence

of $\{a_n\}_{n \geq 1}$

Proof of step 2 - let $P > 0$ be given

$$= \lfloor P \rfloor + 1 \Rightarrow x_{n_{k_0}} > P \Rightarrow x_{n_k} > P \quad \forall k \geq k_0$$

2. HW1

step 1: Find $\{n_k\}_{k \geq 1}$ $n_k < n_{k+1}$ & $x_{n_k} \geq k$

step 2: $x_{n_k} \rightarrow \infty$ as $k \rightarrow \infty$. [$\forall P > 0 \exists k_0 \geq 1$ st $x_{n_k} > P \quad \forall k \geq k_0$]

Proof of step 1 - $M_1 = 1$, as $\{x_n\}_{n \geq 1}$ is unbounded $\exists n_1 \geq 1 : x_{n_1} > 1$

$M_2 = \max\{x_1, \dots, x_{n_1}\} + 1$, as $\{x_n\}_{n \geq 1}$ is unbounded $\exists n_2 \geq 1 : x_{n_2} > 2$

Note $n_2 > n_1$ by definition of M_2

Inductively suppose we have $x_{n_1}, x_{n_2}, \dots, x_{n_k}$ such that $x_{n_i} \geq i$ for $1 \leq i \leq k$

$M_{k+1} = \max\{x_j \mid 1 \leq j \leq n_k\} + 1$ $\{x_n\}_{n \geq 1}$ is unbounded $\exists n_{k+1} > n_k : x_{n_{k+1}} > k+1$

Note $n_{k+1} > n_k$ by definition of M_{k+1}

$\Rightarrow \{x_{n_k}\}_{k \geq 1} : x_{n_k} \geq k \text{ \& } n_{k+1} > n_k$



$\{a_n\}_{n \geq 1}$ be a sequence.

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let $\{n_k\}_{k \geq 1}$ be a strictly increasing

sequence i.e. $n_k < n_{k+1} \forall k \geq 1$

$\{a_{n_k}\}_{k \geq 1}$ is called a subsequence

of $\{a_n\}_{n \geq 1}$

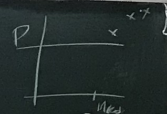
Proof of step 2 - let $P > 0$ be given

$$k_0 = \lfloor P \rfloor + 1 \Rightarrow x_{n_{k_0}} > P \Rightarrow x_{n_k} > P \quad \forall k \geq k_0$$

2. HW1.

step 1: Find $\{n_k\}_{k \geq 1}$ $n_k < n_{k+1}$ & $x_{n_k} \geq k$

$$\boxed{x_{n_{k+1}} > x_{n_k}}$$



step 2: $x_{n_k} \rightarrow \infty$ as $k \rightarrow \infty$. [$\forall P > 0 \exists k_0 \geq 1$ st $x_{n_k} > P \quad \forall k \geq k_0$]

Proof of step 1 - $M_1 = 1$, as $\{x_n\}_{n \geq 1}$ is unbounded $\exists n_1 \geq 1$ $x_{n_1} > 1$

$M_2 = \max\{x_1, \dots, x_{n_1}\} + 1$, as $\{x_n\}_{n \geq 1}$ is unbounded $\exists n_2 \geq 1$ $x_{n_2} > 2$

Note $n_2 > n_1$ by definition of M_2

Inductively suppose we have $x_{n_1}, x_{n_2}, \dots, x_{n_k}$ such that $x_{n_i} \geq i$ for $1 \leq i \leq k$

$M_{k+1} = \max\{x_j \mid 1 \leq j \leq n_k\} + 1$ $\{x_n\}_{n \geq 1}$ is unbounded $\exists n_{k+1} \geq 1$ $x_{n_{k+1}} > k+1$

Note $n_{k+1} > n_k$ by definition of M_{k+1}

$\Rightarrow \{x_{n_k}\}_{k \geq 1}$ $x_{n_k} \geq k$ & $n_{k+1} > n_k$

$$a_n \geq 0$$

$$\sum_{n=1}^{\infty} a_n - \text{Converges.}$$

$$\text{i.e. } T_n = \sum_{k=1}^n a_k$$

$$T_n \rightarrow t \text{ as } n \rightarrow \infty$$

$$t := \sum_{n=1}^{\infty} a_n$$

Partial
sums of
 $\{a_n\}_{n \geq 1}$

limit of
partial
sums
or
series

$$a_k \geq 0 \quad T_n \leq T_{n+1}$$

$$T_n \uparrow t \text{ as } n \rightarrow \infty$$

$$\Rightarrow T_n \leq t \quad \forall n \geq 1$$

$$\{a_{n_k}\}_{k \geq 1}$$

$$\sum_{k=1}^{\infty} a_{n_k} = ?$$

$$n_k < n_{k+1} \quad \forall k \geq 1$$

$$S_m = \sum_{k=1}^m a_{n_k}$$

Q: what happens as $m \rightarrow \infty$
of S_m ?

$$A: \bullet S_m < S_{m+1} \quad \forall m \geq 1$$

$$\left[\because S_{m+1} = \sum_{k=1}^{m+1} a_{n_k} = \sum_{k=1}^m a_{n_k} + a_{n_{m+1}} \right]$$

$$a_{n_{m+1}} \geq 0 \quad \& \quad S_m = \sum_{k=1}^m a_{n_k}$$

$$\Rightarrow S_{m+1} \geq S_m$$

$$\bullet S_m \leq t \quad \forall m \geq 1$$

$$\left[\because S_m = \sum_{k=1}^m a_{n_k} \leq \sum_{n=1}^{n_m} a_n = T_{n_m} \leq t \right]$$

$$\{a_n \geq 0\}$$