

P1 (a)

$$S_n = \sum_{k=1}^n a^k, \quad 0 < a < 1$$

Step 1: $S_n = a \frac{1-a^n}{1-a}$

[Proof of induction]

Step 2: shown that $\lim_{n \rightarrow \infty} a^n = 0$

Step 3: Thm 3.3 (i) $a_n \rightarrow a$ as $n \rightarrow \infty$ then $a_n + b_n \rightarrow a + b$ as $n \rightarrow \infty$
 $b_n \rightarrow b$

(ii) $a_n \rightarrow a$ & c - (constant) then $ca_n \rightarrow ca$ as $n \rightarrow \infty$

In (i) take $a_n = 1 \forall n \geq 1$
 $b_n = -a^n \forall n \geq 1$

$\Rightarrow 1 - a^n \rightarrow 1$ as $n \rightarrow \infty$ & $C = \frac{a}{1-a}$ $\frac{a(1-a^n)}{1-a} \rightarrow \frac{a}{1-a}$ as $n \rightarrow \infty$

$$\lim_{n \rightarrow \infty} S_n = \frac{a}{1-a} \quad \text{D}$$

Second part

$$0 < S_n \quad \forall n \geq 1$$

$$(a^n < 1) \quad 1 - a^n < 1$$

$$\Rightarrow S_n < \frac{a}{1-a} \quad (\text{from step 1})$$

Example $a_n = (-1)^n$

$$a_n = \begin{cases} 3 + \frac{1}{n} & \forall n - \text{odd} \\ 4 + \frac{1}{n} & \forall n - \text{even} \end{cases}$$

$$S_n < S_{n+1} \quad \forall n \geq 1$$

$$\text{Example } \sup_{n \geq 1} \left(-\frac{1}{n} \right) = 0$$

$$\frac{a}{1-a} (1-a^n) \rightarrow \frac{a}{1-a}$$

Fact: $\{S_n\}$

$$- S_n \leq$$

$$\exists M \in \mathbb{R}$$

$$\Rightarrow S_n$$

Proof:

A is bound

Let $S =$

let $\varepsilon > 0$

$$\exists N \geq 1$$

As $S_n \rightarrow m$

$$\forall n > N$$

above. let $s = \sup A \in \mathbb{R}$

we know A is bounded

$$A = \{s_n \mid n \geq 1\}$$

Proof:

$\Rightarrow s_n$ converges to a real number above

bounded

$$\exists M \in \mathbb{R} \quad s_n \leq M \quad \forall n \geq 1$$

$$\forall n \geq 1$$

$$s_n \leq s_{n+1}$$

monotonically increasing

Fact: $(s_n)_{n \geq 1}$

$$N \leq n \quad \forall A \quad S \geq n > 3-S \quad \textcircled{2} \textcircled{3}$$

$$N \leq n \quad \forall A \quad S \geq n \Rightarrow S \geq n \quad \forall A \quad \textcircled{3}$$

$$\textcircled{3} - \quad S_n \leq S_{n+1} \quad \forall n \geq 1 \quad \text{In particular}$$

$$\textcircled{2} \quad S \leq N < 3-S \Rightarrow S \leq N \quad \text{Form } \textcircled{1}$$

$$N \leq n < 3-S \Rightarrow n \geq 1$$

upper bound of A
 $S-3 < S$ so $S-3$ is not an

let $3 > 0$ be given.

$$\textcircled{1} - \quad S_n \leq 8 \quad \forall n \geq 1$$

$n > N$

③

②

Example

$\forall n > N$

$$\text{As } s < s + \epsilon \rightarrow$$

$$s - \epsilon < s_n < s + \epsilon$$

As $\epsilon > 0$ was arbitrary

$$\lim_{n \rightarrow \infty} s_n = s$$

D

also

we know

$$H =$$

Proof.

$$= s_n$$

$\exists M \in \mathbb{R}$

$$- s_n$$

Fact: $d(s_n)$

⑥ Observe let $p > 1$

$$S_n = \sum_{k=1}^n \frac{1}{k^p}$$

$$= S_{n-1} + \frac{1}{n^p}$$

$$\Rightarrow S_n > S_{n-1} \quad \text{①} \quad \forall n \geq 2$$

$$\underline{p > 1} \quad S_{2k+1} = \sum_{i=1}^{2k+1} \frac{1}{i^p}$$

$$S_{2k} = 1 + \sum_{m=1}^k \left[\left(\frac{1}{2m} \right)^p + \left(\frac{1}{2m+1} \right)^p \right]$$



$$S_{2k+1} = 1 + \sum_{i=2}^{2k+1} \frac{1}{i^p}$$

(split even & odd "i")

$$= 1 + \sum_{m=1}^k \left[\left(\frac{1}{2m} \right)^p + \left(\frac{1}{2m+1} \right)^p \right]$$

$$2m+1 > 2m \Rightarrow \frac{1}{2m+1} < \frac{1}{2m} \Rightarrow \left(\frac{1}{2m+1} \right)^p < \left(\frac{1}{2m} \right)^p$$

$$S_{2k+1} < 1 + \sum_{m=1}^k 2 \left(\frac{1}{2m} \right)^p$$

$$\Rightarrow S_{2k+1} < 1 + 2^{1-p} \sum_{m=1}^k \frac{1}{m^p}$$

$$S_{2k+1} < 1 + 2^{1-p} S_k$$

$$\Rightarrow S_{2k+1} < 1 + 2^{1-p} S_{2k+1}$$

$$\Rightarrow (1 - 2^{1-p}) S_{2k+1} < 1$$

$$S_{2k+1} < \frac{1}{1 - 2^{1-p}} \quad \left(\begin{array}{l} p > 1 \\ 1 - 2^{1-p} > 0 \end{array} \right)$$

$$\text{As } S_{2k} < S_{2k+1} \quad \forall k \geq 1$$

$$\Rightarrow S_n < \frac{1}{1 - 2^{1-p}} \quad \forall n \geq 1 \quad \text{--- (2)}$$

From (1) and (2)

So $p > 1$ S_n - converges $\square$

$\left[\begin{array}{l} 2k+1 > 2k \\ S_n \text{ is } \uparrow \text{ sequence} \end{array} \right]$