5 (a)

To show: $S_{n+1} < S_n \quad \forall n \geq 1$

$$S_{n+1} - S_n = \frac{1}{2}\left(\frac{a}{S_n} + S_n\right) - S_n$$

$$= \frac{a - S_n^2}{2S_n} \quad \text{— (1)}$$

$S_{n+1} = \frac{1}{2}\left(\frac{a}{S_n} + S_n\right)$ $\boxed{AM \geq GM}$

$S_1 > \sqrt{a} > 0$

n=1: $S_2 = \frac{1}{2}\left(\frac{a}{S_1} + S_1\right) \geq \left(\frac{a}{S_1} \cdot S_1\right)^{\frac{1}{2}} = \sqrt{a}$

Assume $n=k$ $\quad S_k > \sqrt{a}$

$S_{k+1} = \frac{1}{2}\left(\frac{a}{S_k} + S_k\right)$ is well defined

and $S_k > 0$

$\underset{AM > GM}{\Rightarrow} \quad S_{k+1} > \left(\frac{a}{S_k} S_k\right)^{\frac{1}{2}} = \sqrt{a}$

By induction: $S_n > \sqrt{a} \quad \forall n \geq 1$ — (2)

$\sqrt{a} > 0 \Rightarrow S_n > 0$ — (3)

(2) & (3) $\Rightarrow S_n(S_n) > \sqrt{a}\, S_n$

$\Rightarrow S_n^2 > a$ — (4)

From (1), (4) and (3)

$\Rightarrow S_{n+1} < S_n \quad \forall n \geq 1$ — (5)

& by (2)

$S_n > \sqrt{a} \quad \forall n \geq 1$ — (6)

monotonically decreasing

∴ By (5) and (6) - bounded below

we have, S_n converges to a real number S.

· $S_n \to S$ as $n \to \infty$ & $S_{n+1} - S_n = \frac{a - S_n^2}{S_n}$

$\Rightarrow S_{n+1} \to S$ as $n \to \infty$

5 (a) – Worksheet 2 June 3rd.

let $\alpha_n = S_{n+1} - S_n$

As $S_n \to S$ as $n \to \infty \Rightarrow S_{n+1} \to S$ as $n \to \infty$

[Algebra of limits] $\alpha_n \to S - S = 0$ as $n \to \infty$ → (7)

$\beta_n = \dfrac{a - S_n^2}{2S_n}$

$S_n \geq \sqrt{a} > 0$, $S_n \to S$ as $n \to \infty$, $\Rightarrow \dfrac{1}{S_n} \to \dfrac{1}{S}$

($S = \inf\{S_n | n \geq 1\}$ as $S_n \geq \sqrt{a} \Rightarrow S \geq \sqrt{a}$) >0

(Algebra of limits) $a - S_n^2 \to a - S^2$ as $n \to \infty$

$\Rightarrow \beta_n \to \dfrac{a - S^2}{2S}$ as $n \to \infty$ — (8)

But $\alpha_n = \beta_n \quad \forall n \geq 1$

By (7) & (8) $\quad 0 = \dfrac{a - S^2}{2S}$

$\Rightarrow S^2 = a \Rightarrow S = \sqrt{a}$ □

(3) Worksheet (2) June 3rd, 2019

$\boxed{P=1 \text{ Worksheet 1, June } 6^{th}}$

$a_n = \sum_{k=1}^{n} \dfrac{1}{k}$

$|a_{n+1} - a_n| = \dfrac{1}{n+1}$

As $\dfrac{1}{n+1} \to 0$ as $n \to \infty$

$\Rightarrow |a_{n+1} - a_n| \to 0$ as $n \to \infty$.

Claim:- $\{a_n\}_{n \geq 1}$ is NOT a Cauchy sequence.