Conditional Probability

Ex 210: Suppose you have a fair die and you throw it once. $\Omega = \{1, 2, \dots, 6\}$.

Qn: What is the probability that the outcome is not more than 3?

Ans: $P(\{1,2,3\}) = \frac{3}{6} = \frac{1}{2}$.

Now suppose you are given an extra information that the outcome is an odd number.

Qn: Given this extra information, what is the probability that the outcome is not more than 3?

Ans: Previously,

Outcome :	1	2	3	4	5	6	Total
Prob :	1/6	1/6	1/6	1/6	1/6	1/6	1

Given the extra information,

Outcome :	1	2	3	4	5	6	Total
(Conditional) Prob :	$\frac{1}{3}$	0	$\frac{1}{3}$	0	$\frac{1}{3}$	0	1

(The extra information that another event (the outcome is odd, in this case) has already ocurred, the probabilities of the outcomes have changed. In particular, the probabilities of the outcomes

2, 4, 6 have become 0 and ~~probabilities~~ ~~of~~ the outcomes 1, 3, 5 will remain equally likely and therefore each of these outcomes has prob $\frac{1}{3}$.)

Therefore, ~~$P(\{1,2,3\}) = P(\{1\}) + P(\{2\}) + P(\{3\})$~~

required conditional prob $= \frac{1}{3} + 0 + \frac{1}{3} = \frac{2}{3}$.

[$P(\{1,2,3\} \mid \{1,3,5\}) = P(\{1\} \mid \{1,3,5\}) + P(\{2\} \mid \{1,3,5\}) + P(\{3\} \mid \{1,3,5\}) = \frac{1}{3} + 0 + \frac{1}{3} = \frac{2}{3}$.]

Ex 22: Suppose you have a coin with $P(\text{Head appears in a toss}) = p \in (0,1)$ ~~and~~ and you toss it two times independently. Let $q = 1-p$

Qn: P(Exactly one head appears) = ?

Ans:

Outcome :	HH	HT	TH	TT	Total
Prob :	p^2	pq	pq	q^2	1

Reqd prob $= 2pq$.

Now suppose you are given an extra information that at least one head appears.

Qn: Given this extra information, what is the prob that exactly one head appears?

Ans: Again, the information that at least one head appears, will change the probabilities of all the outcomes. In particular, the prob of the outcome TT will become 0. ~~That~~ Therefore, in order to answer the last question, we have to answer the following question:-

Qn: How will the probabilities of the other outcomes (i.e., HH, HT, TH) change in this case because of the extra information?

~~Ans: We shall change the prob of each of the other outcomes in such a way that proportionately to the~~

Ans: Assume that the new (conditional) prob of each of the outcomes HH, HT, TH given that at least one head appears will be proportional to previous prob of each of the outcomes HH, HT, TH. This implies

Outcome:	HH	HT	TH	TT	Total
Conditional Prob:	$\frac{p^2}{p(p+2q)}$	$\frac{pq}{p(p+2q)}$	$\frac{pq}{p(p+2q)}$	0	1
	$= \frac{p}{2-p}$	$= \frac{1-p}{2-p}$	$= \frac{1-p}{2-p}$	$= 0$	$= 1$

Now, we can answer the previous question.

~~P[~~ Required conditional prob

$$= P\left(\{HT, TH\} \,\middle|\, \{HH, HT, TH\}\right)$$

$$= P\left(\{HT\} \,\middle|\, \{HH, HT, TH\}\right) + P\left(\{TH\} \,\middle|\, \{HH, HT, TH\}\right)$$

$$= \frac{1-p}{2-p} + \frac{1-p}{2-p} = \frac{2-2p}{2-p}.$$

Motivation Behind the Defn of Conditional Prob:

With these two motivating examples, we can now ~~define~~ give the general definition of conditional prob.

Suppose we have a sample space $\Omega = \{\omega_i : i \in I\}$ (where I is countable) and a probability P on Ω. Fix an event $B \in \mathcal{P}(\Omega)$ such that $P(B) > 0$. Then for any event $A \in \mathcal{P}(\Omega)$, we have,

$$P(A \mid B) = \sum_{i \in I:\, \omega_i \in A} P(\{\omega_i\} \mid B) = \sum_{i \in I:\, \omega_i \in A \cap B} P(\{\omega_i\} \mid B)$$

[Since $\omega_i \in B^c \Rightarrow P(\{\omega_i\} \mid B) = 0$]

$$= \sum_{i\in I:\, \omega_i \in A\cap B} \frac{P(\{\omega_i\})}{P(B)}$$ [Since $\omega_i \in B$ $\Rightarrow P(\{\omega_i\}|B) = \frac{P(\{\omega_i\})}{P(B)}$]

$$= \frac{P(A\cap B)}{P(B)}.$$ This calculation motivates the following.

Defn: Suppose $\Omega = \{\omega_i : i \in I\}$ is a ctble sample ~~space~~ space and P is a prob on Ω. Then for any event $B \in \mathcal{P}(\Omega)$ such that $P(B) > 0$, the conditional ~~p~~ probability given the event B is ~~def~~ denoted by $P(\cdot\,|B)$ and is defined ~~by~~ as the set function $P(\cdot\,|B): \mathcal{P}(\Omega) \to \mathbb{R}$ given by

$$P(A|B) = \frac{P(A\cap B)}{P(B)} \quad \text{for all } A \in \mathcal{P}(\Omega).$$

Note: 1. For any event B such that $P(B) > 0$, $P(\cdot\,|B)$ is a probability on Ω, i.e.,

(1|B) $P(E|B) \geq 0$ for all $E \in \mathcal{P}(\Omega)$,

(2|B) $P(\Omega|B) = 1$,

(3|B) $P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i)$ for all pairwise disjoint events $A_1, A_2, \ldots \in \mathcal{P}(\Omega)$.

2. The assumption $P(B) > 0$ ensures that the conditionals prob $P(\cdot \mid B)$ is well-defined. Intuitively this is obvious.

16/08/2013

Motivation of Indepence from Conditional Probability

Fix two events $A, B \in \mathcal{P}(\Omega)$.

We know $$\underbrace{P(A|B)}_{\text{Prob of the event } A \text{ given that } B \text{ has already occurred}} = \frac{P(A \cap B)}{P(B)} \quad \text{if } P(B) > 0.$$

Prob of the event A given that B has already occurred

Therefore, if $P(B) > 0$, then

A is ind of B if $P(A|B) = P(A)$

i.e. if $P(A \cap B) = P(A)\,P(B)$.

On the other hand, if $P(B) = 0$, then obviously, A is ind of B because B never occurs and therefore has no influence on the ~~oc occr~~ occurence / non-occurence of A. Also note that if $P(B) = 0$, then $P(A \cap B) = 0 = P(A)P(B)$.

This motivates us to the following defn: A is ind of B if $P(A \cap B) = P(A)\,P(B)$. Note that

A is ind of B $\Leftrightarrow$ B is ind of A and hence we can say that in this case, A and B are ind.

Exc 26: For two events A, B with ~~$P(A)>0, P(B)>0$~~ $P(A), P(B) \in (0,1)$, TFAE:- (i) A, B are ind, (ii) $P(A|B) = P(A)$ (iii) $P(A|B^c) = P(A)$, (iv) $P(B|A) = P(B)$, (v) $P(B|A^c) = P(B)$.

Properties of Conditional Probability

~~H8~~ Fix a ~~fan~~ ctble sample space Ω, ~~and~~ a prob P on Ω, and an event $B \in \mathcal{P}(\Omega)$ satisfying $P(B) > 0$. Then the conditional prob ~~of~~ $P(\cdot|B)$ given B satisfies the following properties:-

(1|B) $P(E|B) \geq 0 \quad \forall \; E \in \mathcal{P}(\Omega)$.

~~(2|B)~~ Pf: $$P(E|B) = \frac{P(E \cap B)}{P(B)} \geq 0.$$

(2|B) $P(\Omega|B) = 1$

Pf: $$P(\Omega|B) = \frac{P(\Omega \cap B)}{P(B)} = 1.$$

(3|B) $P(\bigcup_{i=1}^{\infty} A_i \mid B) = \sum_{i=1}^{\infty} P(A_i|B)$ for all pairwise disjoint events $A_1, A_2, \dots \in \mathcal{P}(\Omega)$.

Pf: $$P(\bigcup_{i=1}^{\infty} A_i \mid B) = \frac{P((\bigcup_{i=1}^{\infty} A_i) \cap B)}{P(B)} = \frac{P(\bigcup_{i=1}^{\infty} (A_i \cap B))}{P(B)}$$

Since $A_1, A_2, \ldots$ are pairwise disjoint,

so are $A_1 \cap B$, $A_2 \cap B$, $\ldots$.

[Take $1 \leq i < j < \infty$.

Then $(A_i \cap B) \cap (A_j \cap B) = (A_i \cap A_j) \cap B$

$= \emptyset$.]

$$P\left(\bigcup_{i=1}^{\infty} A_i \,\Big|\, B\right) = \frac{P\left(\bigcup_{i=1}^{\infty} (A_i \cap B)\right)}{P(B)}$$

$$= \frac{\sum_{i=1}^{\infty} P(A_i \cap B)}{P(B)}$$

$$= \sum_{i=1}^{\infty} \frac{P(A_i \cap B)}{P(B)}$$

$$= \sum_{i=1}^{\infty} P(A_i | B)$$

<u>Note:</u> 3|B holds as long as $A_1 \cap B, A_2 \cap B, \ldots$ are pairwise disjoint.

4|B $\underline{P(\emptyset | B) = 0}$

<u>Pf:</u> $P(\emptyset | B) = \dfrac{P(\emptyset \cap B)}{P(B)} = \dfrac{P(\emptyset)}{P(B)} = 0$.

<u>Note:</u> In fact $P(A|B) = 0$ for any event $A \subseteq B^c$.

5|B $0 \leq P(E|B) \leq 1 \quad \forall \; E \in \mathcal{P}(\Omega).$

Pf: $0 \leq P(E|B) = \dfrac{P(E\cap B)}{P(B)} \leq 1.$

6|B $P(E^c|B) = 1 - P(E|B) \quad \forall \; E \in \mathcal{P}(\Omega).$

Pf: $P(E^c|B) = \dfrac{P(E^c \cap B)}{P(B)} = \dfrac{P(B) - P(E\cap B)}{P(B)}$

$$= 1 - P(E|B).$$

7|B $P(\bigcup_{i=1}^{n} A_i|B) = \sum_{i=1}^{n} P(A_i|B)$ if the events $A_1\cap B, A_2 \cap B, \ldots, A_n \cap B$ are pairwise disjoint.

(Finite Additivity for Conditional Probability)

8|B If A_1 and A_2 are two events such that $A_1 \cap B \subseteq A_2 \cap B$, then $P(A_1|B) \leq P(A_2|B)$. In particular, if $A_1 \subseteq A_2$, then $P(A_1|B) \leq P(A_2|B)$, i.e., $P(\cdot|B)$ is an increasing set-function.

9|B For any events $A_1, A_2, \ldots, A_n$,

$$P(\bigcup_{i=1}^{n} A_i|B) = \sum_{i=1}^{n} P(A_i|B) - \sum_{1\leq i<j\leq n} P(A_i\cap A_j|B) + \sum_{1\leq i<j<k\leq n} P(A_i\cap A_j \cap A_k|B) - \cdots + (-1)^{n+1} P(A_1 \cap A_2 \cap \cdots \cap A_n|B)$$

(Inclusion Exclusion for Conditional Prob)

10|B For n events $A_1, A_2, \ldots, A_n$,

$$\max_{1\le i\le n} P(A_i|B) \le P(A_1 \cup A_2 \cup \cdots \cup A_n | B) \le \min\left(\sum_{i=1}^{n} P(A_i|B), 1\right).$$

(Boole's ineq for cond prob)

Special Properties of Conditional Prob

11 For events A, B, C with $P(B\cap C)>0$,

$$P(A|B,C) = \frac{P(A\cap B|C)}{P(B|C)} = \frac{P(A\cap C|B)}{P(C|B)}.$$

(• Conditional probabilities can be "conditioned further")

Pf: $P(A|B,C) \coloneqq P(A|B\cap C) = \dfrac{P(A\cap B\cap C)}{P(B\cap C)}.$

$$\frac{P(A\cap B|C)}{P(B|C)} = \frac{P(A\cap B\cap C)/P(C)}{P(B\cap C)/P(C)} = \frac{P(A\cap B\cap C)}{P(B\cap C)}.$$

$$\frac{P(A\cap C|B)}{P(C|B)} = \frac{P(A\cap C\cap B)/P(B)}{P(C\cap B)/P(B)} = \frac{P(A\cap B\cap C)}{P(B\cap C)},$$

•

Note that $P(A\cap B) = P(A|B)\,P(B)$ if $P(B)>0$.

$$P(A\cap B\cap C) = P(A|B,C)\,P(B\cap C) \quad \text{if } P(B\cap C)>0$$
$$= P(A|B,C)\,P(B|C)\,P(C)$$

More generally, we have . . .

$\boxed{12}$ (Chain Rule) For $n (\geq 2)$ events $A_1, A_2, \ldots, A_n$ satisfying $P(A_1 \cap A_2 \cap \ldots \cap A_{n-1}) > 0$,

$$P(A_1 \cap A_2 \cap \ldots \cap A_n) = P(A_n | A_1, A_2, \ldots, A_{n-1}) \, P(A_{n-1} | A_1, A_2, \ldots, A_{n-2}) \cdots P(A_2 | A_1) \, P(A_1).$$

Pf: Very easy — just use induction on n.

For $n = 2$, to prove $P(A_1 \cap A_2) = P(A_2 | A_1) P(A_1)$ if $P(A_1) > 0$ and this is obvious.

Assume that $\boxed{12}$ holds for $n = 2, 3, \ldots, m-1$.

To show $\boxed{12}$ holds for $n = m$ if $P(A_1 \cap A_2 \cap \cdots \cap A_{m-1}) > 0$.

$$P(A_1 \cap A_2 \cap \ldots \cap A_m) \overset{\text{ind}}{\underset{\text{hyp}}{=}} P(A_m | A_1, A_2, \ldots, A_{m-1}) \times P(A_1 \cap A_2 \cap \ldots \cap A_{m-1})$$

$$\overset{\text{ind}}{\underset{\text{hyp}}{=}} P(A_m | A_1, A_2, \ldots, A_{m-1}) \times P(A_{m-1} | A_1, A_2, \ldots, A_{m-2}) \times \cdots P(A_2 | A_1) \, P(A_1).$$

This finishes the proof.

A Simple yet useful application of Chain Rule

Using Chain Rule, we show that all the outcomes are equally likely in SRSWOR.

Recall that in this case if we are drawing a sample of size k from a population of size N, (call them 1, 2, …, N) then the sample space,

$$\Omega = \{(i_1, i_2, \ldots, i_k) : i_1 \in \{1, 2, \ldots, N\}, \text{ for all } j = 2, 3, \ldots, k, \; i_j \in \{1, 2, \ldots, N\} \setminus \{i_1, i_2, \ldots, i_{j-1}\}\}$$

$|\Omega| = {}^N P_k$.

Fix an outcome, $(i_1, i_2, \ldots, i_k) \in \Omega$.

Then $P(\{(i_1, i_2, \ldots, i_k)\})$

$$= P(A_1)\, P(A_2 | A_1) \cdots P(A_k | A_1, A_2, \ldots, A_{k-1}),$$

where A_l is the event that the l^{th} sample point is i_l, $l = 1, 2, \ldots, k$.

Note that $P(A_1) = \frac{1}{N}$,

$P(A_2 | A_1) = \frac{1}{N-1}$,

$$P(A_3 | A_1, A_2) = \frac{1}{N-2},$$

$$\vdots$$

$$P(A_k | A_1, A_2, \ldots, A_{k-1}) = \frac{1}{N-k+1}.$$

Therefore, $P(\{(i_1, i_2, \ldots, i_k)\} = P(A_1 \cap A_2 \cap \ldots \cap A_k]$

$= P(A_1)\, P(A_2 | A_1) \ldots P(A_k | A_1, A_2, \ldots, A_{k-1})$

$= \frac{1}{N} \times \frac{1}{N-1} \cdots \times \frac{1}{N-k+1}$

$= \frac{1}{{}^{N}P_k}$, which shows all the outcomes are equally likely.

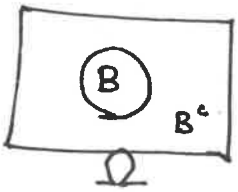
~~Another Application:~~ $\cancel{P(A_2)}$

Remark: If $A_1, A_2, \ldots, A_n$ are mutually ind then chain rule becomes the following product rule: $P(\bigcap_{i=1}^{n} A_i) = \prod_{i=1}^{n} P(A_i)$.

Again, note that for events A, B with $0 < P(B) < 1$, we have

$$P(A) = P(A \cap B) + P(A \cap B^c) = P(B)\,P(A|B) + P(B^c)\,P(A|B^c)$$

Observe that B, B^c forms a partition of Ω.



In other words, B and B^c are disjoint and $B \cup B^c = \Omega$ (B, B^c are "exhaustive").

<u>Def^n:</u> n events $B_1, B_2, \ldots, B_n$ are called exhaustive if $\bigcup_{i=1}^{n} B_i = \Omega$.

<u>Note:</u> A bunch of events are exhaustive if at least one of them occurs.

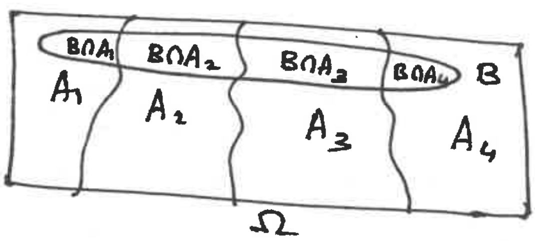
~~$\boxed{13}$ Suppose the events $A_1, A_2, \ldots, A_n$ are pairwise disjoint and exhaustive with $P(A_i) > 0$ for all $i = 1, 2, \ldots, n$.~~

~~Then~~ ~~Also note that~~

13 (Law of Total Probability) Suppose the events $A_1, A_2, \dots, A_n$ are pairwise disjoint and exhaust with $P(A_i) > 0$ for all $i = 1, 2, \dots, n$. Then for ~~and~~ any event $B \in \mathcal{P}(\Omega)$,

$$P(B) = \sum_{i=1}^{n} P(A_i)\, P(B \mid A_i).$$

Note: In this case,



$(n=4)$

$A_1, A_2, \dots, A_n$ form a partition of Ω.

Pf:
$$P(B) = \sum_{i=1}^{n} P(B \cap A_i)$$
[Since $A_1, A_2, \dots, A_n$ a pairwise disjoint ar exhaustive]
$$= \sum_{i=1}^{n} P(A_i)\, P(B \mid A_i)$$

Remark: For pairwise disjoint and exhaustive events $A_1, A_2, \dots$ with $P(A_i) > 0$ for all $i = 1, 2, 3, \dots$
$$P(B) = \sum_{i=1}^{\infty} P(A_i)\, P(B \mid A_i)$$ for any event B.

Back to SRSWOR once again

Let the individuals be $1, 2, \ldots, N$.

Suppose a simple random sample of size k is selecte without replacement.

Qn: What is the prob that 1 is selected in the 2nd d

Ans: Let A_i be the event that i is selected in the 1st draw, $1 \leq i \leq N$.
$A_1, A_2, \ldots, A_N$ is clearly a partition of Ω.
Let B be the event that 1 is selected in the 2nd draw.

Then $P(B|A_1) = 0$

and $P(B|A_2) = P(B|A_3) = \cdots = P(B|A_N) = \frac{1}{N-1}$

Also $P(A_1) = P(A_2) = \cdots = P(A_N) = \frac{1}{N}$.

Therefore, by Law of Total Probability,

$$P(B) = \sum_{i=1}^{N} P(B|A_i)\,P(A_i)$$

$$= \left(0 \times \frac{1}{N}\right) + \underbrace{\left(\frac{1}{(N-1)} \times \frac{1}{N}\right) + \left(\frac{1}{N-1} \times \frac{1}{N}\right) + \cdots + \left(\frac{1}{N-1} \times \frac{1}{N}\right)}_{(N-1) \text{ times}}$$

$$= \frac{1}{N}.$$

Similarly, we can show that the prob that 1 is selected in the j^{th} draw $= \frac{1}{N}$ for $j = 3, 4, \ldots, N$.

Exc 27: Show the above statement for $j = 3$ using the law of total probability.

[14] (Bayes Rule) Suppose $A_1, A_2, \ldots, A_n$ are pairwise disjoint and exhaustive events with $P(A_i) > 0$ for all $i = 1, 2, \ldots, n$. Then for any event B ~~with~~ with $P(B) > 0$, and for all $j = 1, 2, \ldots, n$, we have

$$P(A_j \mid B) = \frac{P(B \mid A_j)\, P(A_j)}{\sum_{i=1}^{n} P(B \mid A_i)\, P(A_i)}$$

Pf:

$$P(A_j \mid B) = \frac{P(B \cap A_j)}{P(B)}$$

$$= \frac{P(B \mid A_j)\, P(A_j)}{\sum_{i=1}^{n} P(B \mid A_i)\, P(A_i)}$$

Application of Bayes Rule : Again back to SRSWOR.

Qn: Suppose that 1 was selected at the 2nd draw. ~~was~~ What is the (Conditional) prob that 2 was selected at the 1st draw?

Ans: Recall

$B \equiv 1$ was selected at the 2nd draw

$A_i \equiv i$ was selected at the 1st draw, $1 \leq i \leq N$

Then by Bayes Rule,

$$P(A_2 \mid B) = \frac{P(B \mid A_2)\, P(A_2)}{\sum_{i=1}^{N} P(B \mid A_i)\, P(A_i)}$$

$$= \frac{\frac{1}{N-1} \times \frac{1}{N}}{0 \times \frac{1}{N} + (N-1) \times \frac{1}{N-1} \times \frac{1}{N}} = \frac{1}{N-1}.$$

Similarly,

$$P(A_3 \mid B) = P(A_4 \mid B) = \cdots = P(A_N \mid B) = \frac{1}{N-1}.$$

Also $$P(A_1 \mid B) = \frac{P(A_1 \cap B)}{P(B)} = 0.$$

Remark: This shows that the roles of 1st and 2nd draws are exchangeable in nature. In general, all the draws are exchangeable.

Composite Experiments: These experiments are described by specifying certain conditional probabilities. In other words, the probabilities of the outcomes in the sample space are not given to us directly – they have to be derived from the specified conditional probabilities.

e.g., SRSWOR, Urn Models.

Ex 23: (Polya's Urn Scheme) $b, r, c \in \mathbb{N}$.

b black balls
r red balls

- Choose a ball at random.
- Look at its colour.
- Put the ball back.
- Add c more balls of the same colour.
- Repeat this procedure again and again.

Let

$R_i \equiv$ "i^{th} ball drawn is red", $i = 1, 2, \ldots$.

$$P(R_2) = P(R_2 \mid R_1)\, P(R_1) + P(R_2 \mid R_1^c)\, P(R_1^c)$$

$$= \frac{r+c}{b+r+c} \cdot \frac{r}{b+r} + \frac{r}{b+r+c} \cdot \frac{b}{b+r}$$

$$= \frac{r}{b+r} = P(R_1)$$

[So, on the avg, we are not changing the proportions.

$$P(R_2^c) = \frac{b}{b+r}$$

$$P(R_1|R_2) = \frac{P(R_2|R_1)\,P(R_1)}{P(R_2)} = P(R_2|R_1)$$

(The roles of 1st and 2nd draw are exchangeable.)

Suppose ~~We make exa~~ $E_{n_b,n}$ is the event that exactly n_b many black balls are drawn in the first n draws, $(n_b \leq n)$. Let $n_r = n - n_b$.

~~Then~~ P[First n_b balls are black and the next n_r balls are red]

$$= \frac{b(b+c)\ldots(b+n_b c-c)\quad r(r+c)\ldots(r+n_r c-}{(b+r)(b+c+r)\ldots(b+r+n_b c-c)(b+r+n_b c)\ldots(r+b+nc-}$$

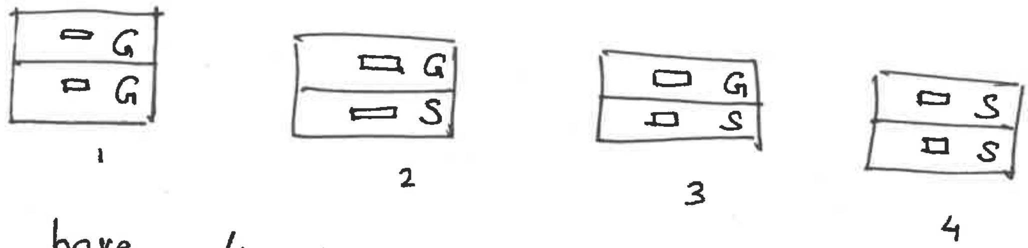
Note that for any specific arrangement of n_b black and n_r red balls, the above prob remains the sa (The denominator is same and the numerator is obtained with a different rearrangement of the same factors.)

~~Therefore~~ $\cancel{P(E_{n_b,n}) = \binom{n}{n_b} \frac{b(b+c)\ldots(b+n_b c-c)\, r(r+c)\ldots}{(b+r)(b+r+c)+\cdots+(b+r+n}}$

Therefore,

$$P(E_{n_b,n}) = \binom{n}{n_b} \frac{b\,(b+c)\ldots(b+n_b c-c)\; r\,(r+c)\cdots(r+n_r c-c)}{(b+r)\,(b+r+c)+\ldots+(b+r+nc-c)}$$

Ex 24:



Suppose you have 4 chests of 2 drawers each with gold / silve coins as shown in the above picture.

You choose a chest at random, ~~and~~ then choose a drawer at random and find out that it has a silver coin. What is the prob that the other drawer in that chest has a gold coin?

Let $C_i \equiv$ Chest # i is chosen, $i = 1, 2, 3, 4$.

$S \equiv$ The ~~first~~ coin in the selected drawer is a silver coin.

$P(S|C_1) = 0$, $P(S|C_2) = \frac{1}{2}$, $P(S|C_3) = \frac{1}{2}$, $P(S|C_4$

(127)

$P(C_i) = \frac{1}{4} \quad \forall\ i = 1, 2, 3, 4. \Rightarrow P(S) = \frac{1}{8} + \frac{1}{8} + \frac{1}{4} = \frac{1}{2}$

Required prob $= P(C_2 \cup C_3 \mid S)$

$$= P(C_2 \mid S) + P(C_3 \mid S)$$

$$= \frac{P(S \mid C_2)\,P(C_2)}{P(S)} + \frac{P(S \mid C_3)\,P(C_3)}{P(S)}$$

$$= \frac{\frac{1}{2} \times \frac{1}{4}}{\frac{1}{2}} + \frac{\frac{1}{2} \times \frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}.$$