Mathematical Induction – An Impresario of the Infinite

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In the natural sciences, if a certain phenomenon is observed to occur a number of times, often a general law is formulated. This process is called *empirical induction.* In general, any reasoning that draws a general conclusion based on verification of particular cases is known as induction. But, in mathematics, a statement involving a natural number n might turn out to be erroneous even if it happens to be true for the first ten, or thousand, or even million natural numbers. For instance, the numbers $2^{2^0}+1=3$, $2^{2^1}+1=5$, $2^{2^2}+1=17$, $2^{2^3}+1=257$, $2^{2^4}+1=65537$ are all prime numbers and the 17th century mathematician Pierre de Fermat suggested that $2^{2^n}+1$ must be prime for every positive integer n. However, a century later, another great mathematician Leonhard Euler showed that $2^{2^5}+1=641\times 6700417$. An even more convincing example is the following. If we evaluate the expression $991n^2+1$ for small values of n, the resulting number is not the square of a whole number. But, for $n=12055735790331359447442538767$, the value is a perfect square. Indeed, this is the smallest value of n for which it is a square! This tells us that, in mathematics, a lot of care is needed to establish an induction procedure which proves a mathematical theorem for each of an infinite sequence of cases, without exception. The method of mathematical induction is such a procedure. Let us start with a simple example.

Suppose we want to prove the statement that $2^n > n$ for every natural number n. Clearly, this inequality holds for $n=1$. Now, to prove the inequality for all natural numbers, we consider an *arbitrary* natural number $k \geq 1$. We *assume* that the inequality $2^k > k$ holds. Then, for the *next* natural number $k+1$, $2^{k+1}=2\times 2^k > 2k$ by our *assumption* that $2^k > k$. Now, $2k=k+k \geq k+1$, so that the inequality $2^{k+1} > k+1$ follows. Thus, we have proved that *if* the inequality is true for any particular k, *then* it is also true for $k+1$.

The crux of the above argument rests on the points:

(0) Given an infinite sequence of statements $P_r, P_{r+1}, \cdots$ we would like to prove that there is a 'next' to any statement, and each particular statement can be reached in a finite number of steps starting from the 'first' statement P_r.

(1) There is a general method of proving that for any $n \geq r$, *if* P_n is true, *then* P_{n+1} is true; and

(2) The first statement P_r is *known* to be true.

It is believed that these rules of logic are as fundamental to mathematics as the classical rules of Aristotelian logic.

It is necessary to verify both the steps (1) and (2) to avoid landing in absurdities. For example, if step (2) that 'starts induction' is not verified, one can 'prove' that all natural numbers are equal as follows. For, simply denote by P_n the statement '$n = n + 1$'. Then, obviously, if P_n is assumed to be true, then $n = n + 1$ and so $n + 1 = n + 2$, which means that P_{n+1} is also true.

Everybody has seen instances of mathematical induction being applied. The summing of arithmetic and geometric progressions are usually done by this method.

An important point is in order here. *Mathematical induction can be used to prove a statement that is given to begin with. As for coming up with that statement itself (as a guess, say), it is altogether a different matter. Therein lies the creative element which cannot be pinned down by any general rules.*

As we observed earlier, mathematical induction is a procedure that involves such extremely 'believable' logic that we accept it as valid reasoning. But, interestingly, we can actually prove its validity if we assume another believable principle which is that *any non-empty set of positive inte-*

gers has a least number. That this principle gives a proof of the validity of mathematical induction is left as an exercise to the reader.

We now proceed to give instances of various guises under which the method of mathematical induction appears and proves fruitful.

The following is a slight variant of the form in which induction is used.

To prove an infinite sequence $P_k, P_{k+1}, \cdots$, of assertions, one verifies the two steps:

(i) P_k is true.

(ii) For any $n \geq k$, if we assume that all the $P_k, P_{k+1}, \cdots, P_n$ hold good, then P_{n+1} also holds true.

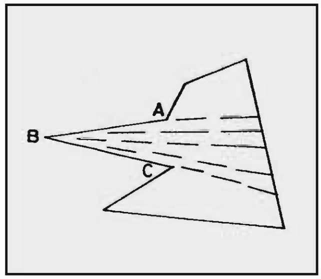


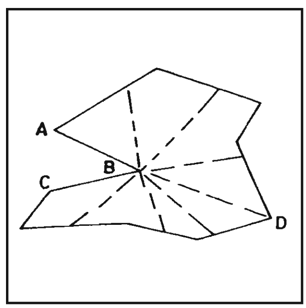
Figure 1.

Induction in Geometry

As an example, let us show that the sum of the interior angles of a (not necessarily convex) polygon of n sides is $180(n-2)$ degrees for all $n \geq 3$. Call this statement P_n. P_3 is true as the sum for a triangle is 180 degrees. P_4 is also true since any quadrilateral can be split into two triangles.

Now, let $n > 4$ and we assume that P_k is true for $k = 3, 4, \cdots, n-1$. Let $A_1, A_2, \cdots, A_n$ be the vertices of a polygon with n sides. We first notice that there is always a diagonal (i.e., a segment A_iA_j that is not a side) that splits the polygon into two with smaller numbers of sides. To see this, consider three neighbouring vertices A, B, C. Consider all the rays emanating from B and filling the interior angle ABC. We terminate any ray when it first meets a side or vertex of the polygon. Either all these rays intersect only one side (*Figure 1*) or they intersect more than one side (*Figure 2*). In the first case, AC is a diagonal that splits the original polygon into a triangle and a polygon with $n-1$

Figure 2.



[1] That we have interchanged the roles of girls and boys in our version is a miracle of modern genetics!

sides. In the second case at least one ray terminates on a vertex other than A or C. Call such a vertex D. Then, BD is a diagonal splitting the polygon into two with smaller numbers of sides.

Therefore, in general, let A_1A_k denote a diagonal which splits the polygon $A_1A_2 \cdots A_n$ into the polygons $A_1A_2 \cdots A_k$ and $A_kA_{k+1} \cdots A_nA_1$ of k and $n-k+2$ sides respectively. By the induction hypothesis, P_k and P_{n-k+2} are true i.e., the sum of the interior angles of the original polygon $A_1A_2 \cdots A_n$ is $180(k-2)+180(n-k) = 180(n-2)$ degrees. So, P_n is true, which proves by induction that P_r is true for every $r \geq 3$.

After this standard example, we look at an example where it may not be quite apparent that induction can be used.

The Marriage Problem

The classical 'marriage problem' can be stated as follows. Suppose that each of a set of girls is acquainted with a subset from a given set of boys. Is it possible for each girl to marry one of her acquaintances? [1] Obviously, a necessary condition is that every set of m girls be collectively acquainted with at least m boys. *That this suffices is the assertion.* Here is a proof by induction.

Let n denote the number of girls. If $n = 1$, the assertion is trivial. If $n > 1$ and if it is true that every set of m girls, $1 \leq m < n$, has at least $m+1$ acquaintances, then an arbitrary girl is allowed her choice and the rest are referred to the induction hypothesis. If, on the other hand, some group of m girls, $1 \leq m < n$, has precisely m collective acquaintances, then this set of m girls is married off by induction and, it is indeed true that the rest of the $n-m$ girls satisfy the necessary condition with respect to the remaining boys. If this were not so, then some set of s spinsters with $1 \leq s \leq n-m$ know fewer than s bachelors, and this set of s spinsters together with the m just-married girls would have known fewer than $s+m$ boys.

The reader is invited to apply induction to solve the following two problems:

Exercise : (Consecutive Number Problem)

Agatha and Beula are 'given' two consecutive natural numbers n and $n+1$. Both know that the numbers are consecutive but neither knows whose number is bigger. After every minute a beep is heard and each is asked to simultaneously say out aloud whether she knows the other's number. Prove by induction on the smaller number n that the person who has the number n guesses correctly after precisely the n-th beep.

Exercise : (Macaulay Expansion)

Given a natural number $d \geq 2$, let us write down the d-tuples of positive integers in a strictly decreasing order. Order the tuples lexicographically. Prove that the number of tuples appearing prior to a particular tuple $(k_d, k_{d-1}, \quad, k_1)$ is precisely $\binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \cdots \binom{k_1}{1}$.

This proves that any n has an unique expansion $n = \binom{k_d}{d} + \binom{k_{d-1}}{d-1} + \quad \binom{k_1}{1}$ where $k_d > k_{d-1} > \quad \cdot k_1$. Here $\binom{n}{r}$ denotes the binomial coefficient which is 0 when $n < r$.

Induction incognito - Use of a 'Dummy' Element

Look at the following statement:

If $a_1 < a_2 < \quad < a_{n+1}$ are integers from the set $\{1, 2, . \quad, 2n\}$, then a_i divides a_j for some $i < j$.

This can be solved by the 'pigeon-hole principle' as follows. Write $a_i = 2^{k_i} l_i$ with l_i odd. Then, $l_1, \quad, l_{n+1}$ being $n+1$ odd numbers between 1 and $2n$ cannot be different. If

$l_i = l_j = l$ with $i < j$, then, clearly $a_i = 2^{k_i}l$ divides $a_j = 2^{k_j}l$.

In terms of economy and elegance, this is unbeatable. However, we find to our surprise that even induction works and, in fact, proves the following more general statement:

$\boxed{\text{S}}$ *Let $r \geq 1$, and let $A \subset \{1, 2, \quad, 2^r n\}$ be a subset of cardinality $(2^r - 1)n + 1$. Then, there exists a chain of $r + 1$ elements of A with each dividing the next.*

Let us prove the original statement (corresponding to $r = 1$). Note that it is clearly true for $n = 1$. Assume it is true for n. Consider now $n + 2$ numbers $a_1 < \quad < a_{n+2}$ among 1 to $2n + 2$. If $a_{n+1} \leq 2n$, we are done by the induction hypothesis. In the contrary case, we must have $a_{n+1} = 2n + 1$ and $a_{n+2} = 2n + 2$. If one of the a_i's is $n + 1$, we are done as it divides a_{n+2}. So, suppose $a_i \neq n + 1$ for any i. We may also assume that none of the n numbers $a_1, . \quad, a_n$ divides another or else we have nothing to prove. Now, we *put in this new number* $n + 1$ (as a 'dummy element') to get $n + 1$ numbers between 1 and $2n$. By induction hypothesis, one of these $n + 1$ numbers divides another. Since this has happened only after the advent of the new number $n + 1$, it must be that either: (i) some a_i $(i \leq n)$ divides $n + 1$ or (ii) $n + 1$ divides some a_i $(i \leq n)$. But, clearly (ii) can not happen as $n + 1 \neq a_n \leq 2n$. Thus, some a_i $(i \leq n)$ divides $n + 1$ and, therefore, divides $2n + 2 = a_{n+2}$ also. Thus, we used $n + 1$ as a 'dummy element' in this proof.

The reader is urged to complete the proof of the general statement along the same lines.

Now, we come to a final example where induction appears in a different guise.

Backward Induction

If a statement is easily proved for a *particular* infinite subsequence of positive integers, it might be worthwhile to try whether 'backward induction' works. By this, we mean the following. Suppose we want to prove statements P_n for all positive integers n. Suppose, further, it is easy to check the veracity of P_n for all n in an infinite sequence of natural numbers. Then, if we check that for any $m \geq 2$, the truth of P_m implies the truth of P_{m-1}, the statements P_n follow for all positive integers n.

An instance is the familiar arithmetic mean – geometric mean inequality

$$P_n \quad : \quad (\sum_{i \leq n} a_i)^n \geq n^n \prod_{i \leq n} a_i$$

for arbitrary non-negative real numbers a_i, where equality holds if, and only if, all the numbers are equal.

On the one hand, we prove this for $n = 2^k$ by induction on k. Let $k = 1$. Then, $(a_1+a_2)^2 \geq 4a_1a_2$ with equality exactly when $a_1 = a_2$, since the difference $(a_1+a_2)^2 - 4a_1a_2 = (a_1 - a_2)^2$. Assume that P_n is true for $n = 2^r$ $r \leq k$. Let a_i, $i \leq 2^{k+1}$, be non-negative real numbers. Then, $\sum_{i \leq 2^{k+1}} a_i = \sum_{i \leq 2^k} b_i$ where $b_i = a_{2i-1} + a_{2i}$. Therefore,

$$(\sum_{i \leq 2^{k+1}} a_i)^{2^{k+1}} = (\sum_{i \leq 2^k} b_i)^{2^{k+1}} = ((\sum_{i \leq 2^k} b_i)^{2^k})^2$$

$$\geq ((2^k)^{2^k} \prod_{i \leq 2^k} b_i)^2 = 2^{k2^{k+1}} \prod_{i \leq 2^k} b_i^2 \geq 2^{k2^{k+1}} \prod_{i \leq 2^k} (4a_{2i-1}a_{2i})$$

$$= 2^{k2^{k+1}} 4^{2^k} \prod_{i \leq 2^{k+1}} a_i = 2^{(k+1)2^{k+1}} \prod_{i \leq 2^{k+1}} a_i$$

which proves that $P_{2^{k+1}}$ is true. Hence, by induction, P_{2^r} is valid for all $r \geq 1$. Moreover, note that the above proof also shows that equality $(\sum_{i \leq 2^{k+1}} a_i)^{2^{k+1}} = 2^{(k+1)2^{k+1}} \prod_{i \leq 2^{k+1}} a_i$ implies that all the inequalities occurring on the way are equalities which again proves by induction that equality can hold in P_{2^r} if, and only if, all the a_i are equal.

On the other hand, for any m, the validity of P_m implies the validity of P_{m-1} as follows. Let $a_1, \cdots, a_{m-1}$ be given. Consider $a_m = \frac{1}{m-1}\sum_{i\leq m-1} a_i$. Then,

$$(\sum_{i\leq m-1} a_i)^m = (\frac{m}{m-1})^m(\sum_{i\leq m-1} a_i)^m$$

$$\geq (\frac{m}{m-1})^m(m-1)^{m-1}(\prod_{i\leq m-1} a_i)(\sum_{i\leq m-1} a_i) = m^m \prod_{i\leq m} a_i\,.$$

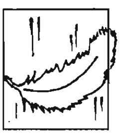
Once again, by induction, equality implies that all the numbers are equal.

To end our discussion, the reader is invited to apply induction on the positive integer p below to prove the following result which solves an interesting two-player game called Euclid.

Let (p, q) be a pair of positive integers satisfying $p > q$. Each player subtracts a multiple of the smaller number from the bigger one without making the result negative. The winner is the one first hitting the highest common factor of p and q. Then, there is a winning strategy for the first player if, and only if, $q < \frac{1}{2}(\sqrt{5}-1)p$.

Suggested Reading

- R Courant and H Robbins. *What is Mathematics?*. Oxford University Press, 1941.
- L I Golovina and I M Yaglom. *Induction in Geometry*. Little Mathematics Library. Mir Publishers. Moscow, 1979.



Hot Water Freezes Faster!

How is it possible for hot water to freeze more quickly than cold? This peculiar phenomenon, first noticed by Aristotle in the 4th century BC, has baffled scientists for generations. The phenomenon is today known as the Mpemba effect, after the Tanzanian schoolboy Erasto Mpemba. In the 1960s, Mpemba became a laughing stock after telling his science teacher he could make ice-cream mixture freeze faster by warming it before putting it into the freezer.

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