Problems Due Monday June 17th: 1,2,6,9,10,11,12,16

1. Suppose $X \sim Bin(n, p)$. Compute $E(X)$ and $Var(X)$ directly from definition as described in the class. You may use the hints given in the class for the latter.

2. If X_1, X_2 and X_3 are three independent discrete random variables each taking finitely many values, then show that X_1 and X_2 are independent.

3. Suppose X_1 and X_2 are two discrete random variables (not necessarily independent) each taking finitely many values.

   (a) If X_1 and X_1 are independent, then show that they are uncorrelated (that is, $Cov(X_1, X_2) = 0$).

   (b) Give an example to show that the converse of the above does not hold. Justify all your steps.

4. If $X \sim$ Geometric(p) (as defined in the class), then show from definition that $E(X) = 1/p$.

5. Consider Polya's Urn Scheme as described in the class. Suppose X_i is the indicator that the i^{th} ball drawn in black. Show that for each $n \in \mathbb{N}$, the random vector $(X_1, X_2, \ldots, X_n)$ is exchangeable. **(Hint: You have to use Chain Rule. Solve it for $n = 3$ case first, and then think how to write down the argument in the general case.)**

   [This exchangeability property explains various observations made by us in the first lecture. For example, one can now show, without using a complicated induction, that for each $n \in \mathbb{N}$,

$$P(R_n) = P(X_n = 0) = P(X_1 = 0) = P(R_1) = \frac{r}{b+r}.$$

   The notations are as in the first lecture.]

6. Fix $M, N \in \mathbb{N}$ such that $M < N$. Suppose you have items which are numbered $1, 2, \ldots, N$. Suppose also that the first M of these are defective and the rest are not. Suppose a random sample of size $n\,(< N)$ is chosen *without replacement* from these N items. Let X_i denote the indicator that the i^{th} drawn item (of your sample) is defective. Let X be the total number of defective items in the sample.

   (a) Using Chain Rule, show that the random vector $(X_1, X_2, \ldots, X_n)$ is exchangeable.

   (b) Using (a) and Thm 1 (as done in the second lecture), compute $E(X)$.

   (c) Using (b) and Thm 5 (as done in the second lecture), compute $Var(X)$.

   (d) In order to solve (c), you had to compute $Cov(X_i, X_j)$ for each $i < j$. Is this covariance positive or negative? Justify your answer algebraically and also intuitively.

   [The above random variable X is said to follow hypergeometric distribution with parameters N, M, n (denoted by $X \sim Hyp(N, M, n)$). Please search the phrase "Mark and Recapture" online and read from Wikipedia to know about application of this probability distribution in estimation of the size of an animal population. In particular, the above calculations do have real-life applications.]

7. **(First Challenging Problem)** Suppose r distinguishable balls are arranged at random in n boxes (each with unlimited capacity) numbered $1, 2, \ldots n$. Let N denote that number of empty boxes. Using the "indicator method", compute $E(N)$ and $Var(N)$.

   *[The above is a toy model for the **Maxwell-Boltzmann Statistics** arising in Physics. In the actual model, the balls need to be replaced by particles, and the boxes need to be replaced by cells (typically determined by energy-levels or other physical characteristics). Therefore, this calculation too is important in Statistical Physics. There is another related model (where particles/balls are considered indistinguishable) known an **Bose-Einstein Statistics**.]*

8. **(Second Challenging Problem)** Consider the "top to random shuffle" of a pack of 52 cards as shown in the third lecture. Let N denote the number of shuffles needed for the initial bottommost card to come to the top. Compute $E(N)$ and $Var(N)$ using Thm 1 and Thm 3 (both as in the second lecture), respectively.

   [It can be shown that after $N+1$ steps, the pack of cards gets mixed very well, i.e., all the permutations of 52 cards become approximately equally likely. That is why $E(N+1) = E(N)+1$ and $Var(N+1) = Var(N)$ are important quantities.]

9. Let S be the set of limit points of the sequence $\{x_n\}_{n=1}^{\infty}$. In each of the cases below find L:

   (a) $x_n = 1 + (-1)^n$

   (b) $x_n = \frac{1}{n}$

   Note: you need to identify elements in S and also show a real number not in S is not a limit point.

10. Let a be a limit point of a sequence $\{x_n\}_{n=1}^{\infty}$. Show that there is a subsequence of $\{x_{n_k}\}$ that converges to a.

11. Suppose $\{x_n\}_{n=1}^{\infty}$be a sequence of real numbers and let E its set of limit points. Show that $E = \{x\}$ for some $x \in \mathbb{R}$ if and only if $\{x_n\}_{n=1}^{\infty}$converges to x.

12. Let W_1 and W_2 be subspaces of V given by

$$W_1 = \{A \in M_2(\mathbb{R}) \mid A = A^{\text{tr}}\}$$

   and

$$W_2 = \{A \in M_2(\mathbb{R}) \mid EAE = A^{\text{tr}}\}$$

   where $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Find the dimensions of $W_1, W_2, W_1 + W_2$ and $W_1 \cap W_2$ and find a basis in each case.

13. Let A be an $m \times n$ matrix with rank m prove that $m \leq n$ and there exists an $n \times m$ matrix B such that $AB = I_m$.

14. (a) Let V be an n dimensional vector space over $\mathbb{Z}_p$ (finite field with p-elements). For any $m \leq n$, find the number of m-dimensional subspaces of V.

    (b) For any $n \in \mathbb{N}$ and prime p, determine the cardinality of $\mathrm{GL}_n(\mathbb{Z}_p)$ (the group of invertible matrices with entries from the finite field with p-elements).

15. Let $A, B \in M_n(\mathbb{R})$ be such that $AB = -BA$. Prove that if n is odd then either A or B is not invertible.

16. Let T be the linear operator on $\mathbb{R}^2$, the matrix of which in the standard ordered basis is

$$\begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}.$$

    (a) Prove that the only subspaces of $\mathbb{R}^2$ invariant under T are $\mathbb{R}^2$ and the zero subspace.

    (b) If U is the linear operator on $\mathbb{C}^2$, the matrix of which in the standard ordered basis is A, show that U has 1-dimensional invariant subspaces.