G stands for a group.
$\mathbb{Z}$ denotes the group of integers under addition, and $\mathbb{Q}$ denotes the group of rational numbers under addition
S_n denotes the symmetric group of degree n, ie. the group of permutations on n elements.

1. Let $G = \{z \in \mathbb{C} \mid z^n = 1 \text{ for some } n \in \mathbb{Z}^+\}$. Prove that for any fixed integer $k > 1$ the map from G to itself defined by $z \mapsto z^k$ is a surjective homomorphism but is not an isomorphism.

2. Find out all homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}$. Which ones are injective, which ones surjective? Deduce the number of automorphisms of $\mathbb{Z}$. (An *automorphism* of a group G is an isomorphism from G to itself.)

3. Prove that for each fixed nonzero $c \in \mathbb{Q}$ the map from $\mathbb{Q}$ to itself defined by $q \mapsto cq$ is an automorphism of $\mathbb{Q}$.

4. Fix an element $g \in G$. Denote by ϕ_g the map:

$$\begin{array}{rcl} \phi_g : G & \longrightarrow & G \\ x & \mapsto & gxg^{-1} \end{array}$$

for all $x \in G$. Show that ϕ_g is an automorphism of G for all $g \in G$ (called conjugation by g). What is the inverse map of ϕ_g?