1. Prove that there does not exist a linear map from $\mathbb{F}^5$ to $\mathbb{F}^2$ whose null space equals

$$\{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{F}^5 \mid x_1 = 3x_2 \text{ and } x_3 = x_4 = x_5\}.$$

2(a) If A is a linear transformation of rank one, then there exists a unique scalar α such that $A^2 = \alpha A$.

2(b) If $\alpha \neq 1$, then $1 - A$ is invertible.

3. Let $T : P_2(R) \to P_2(R)$, where $P_2(R)$ is set of real polynomials of degree less than equal to two, be a linear map defined by,

$$T(f(x)) = f(1) + f'(0)x + (f'(0) + f''(0))x^2$$

where $f'(x)$ denotes the derivative of $f(x)$.

(a) Find characteristic polynomial of T.

(b) Prove that T is diagonalizable.

(c) Find a basis β of $P_2(R)$ such that $[T]_\beta$ is a diagonal matrix.

4. Let $T : V \to V$ be a linear operator on a vector space of dimension two. Assume that T is not multiplication by a scalar. Prove that there is a vector $v \in V$ such that $(v, T(v))$ is a basis of V, and describe the matrix of T with respect to that basis.