1. We shall define the discrete Fourier transform for bi-infinite sequences in the following manner.

$$\{ \quad a_t \quad : \quad t = \ldots, -1, 0, 1, \ldots \} \longleftrightarrow \{ \quad A(f) \quad : \quad -\frac{1}{2} < f < \frac{1}{2} \},$$

The discrete Fourier transform for $\{a_t : t \in \mathbb{Z}\}$ is given by[1]

$$A(f) \equiv \sum_{t=-\infty}^{\infty} a_t e^{-i2\pi f t}, \qquad -\infty < f < \infty$$

1. Show that:

(a) $A(\cdot)$ is periodic with unit period. I.e. Show that

$$A(f + j) = A(f), \text{ for all } j.$$

(b) if a_t is real-valued, then $A(-f) = A^*(f)$

(c) if a_t is periodic with period N then observe that

$$A(f) = A(f + 2\pi N).$$

Conclude that it is enough to sum over one period of a to work with the Discrete Fourier transform (as for finite sequences) with

$$A_k = \sum_{t=0}^{N-1} a_t e^{ik\frac{2\pi}{N}t}.$$

[1]Can you come up with a precise definition for the convergence of series in the definition of discrete Fourier transform ?

2.Compute the discrete-time Fourier transform of the following sequence:

(a) $a_t = \begin{cases} \frac{1}{4^t} & t \geq 0, t \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$

(b) $a_t = \begin{cases} 1 & t \in \{0, 1, 2, 3, 4\} \\ 0 & \text{otherwise} \end{cases}$

3. Let $m, N \in \mathbb{N}$. Consider the periodic discrete-time exponential sequence given by

$$a_t = \exp(i\frac{2\pi mt}{N}) \quad \text{for } t \in \mathbb{Z}.$$

Find the period of a.

4. Let $\{a_t : t \in \mathbb{Z}$ be such that $a_t = 0$ for $t \notin \{0, 1, 2, \ldots N-1\}$. Let $\{A(f) : -\frac{1}{2} < f < \frac{1}{2}\}$ be its discrete Fourier transform. We generate the periodic sequence

$$y_t = \sum_{r=-\infty}^{\infty} a_{t+rN}.$$

(a) Find the period of y.

(b) Write an expression for the discrete Fourier transform $\{B_k : K = 0, 1, \ldots N-1\}$ in terms of a.

(c) Write an expression relating B to A.