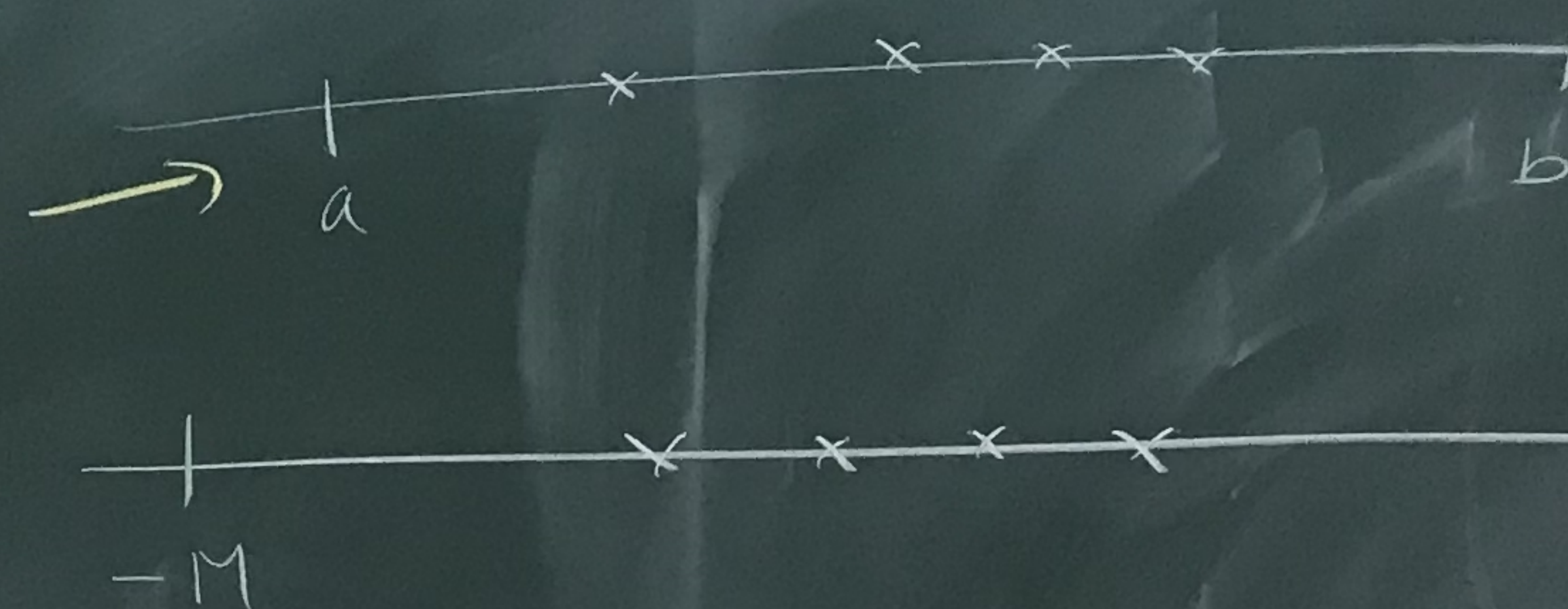


1 (a) Picture A is bounded

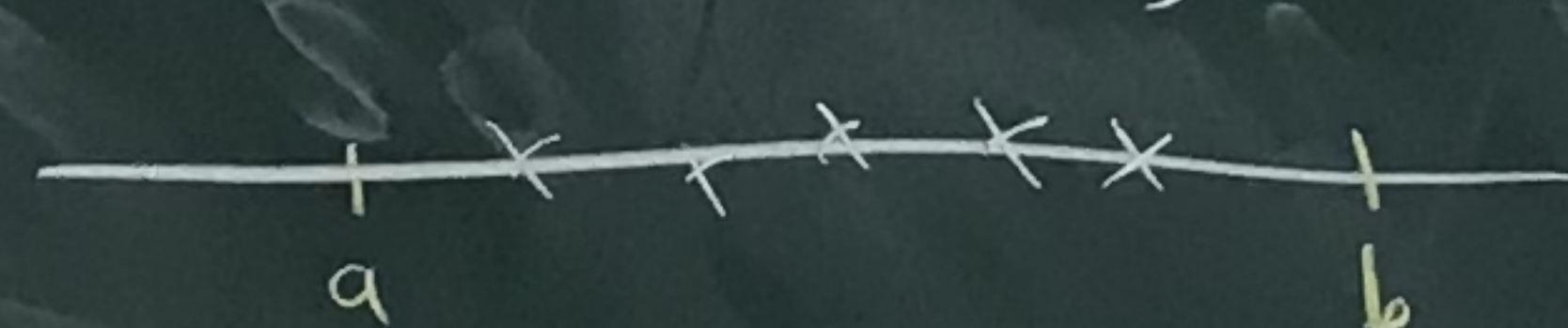
(*) (*) $(\exists a, b \in \mathbb{R} \text{ such that } a \leq x \leq b \forall x \in A)$



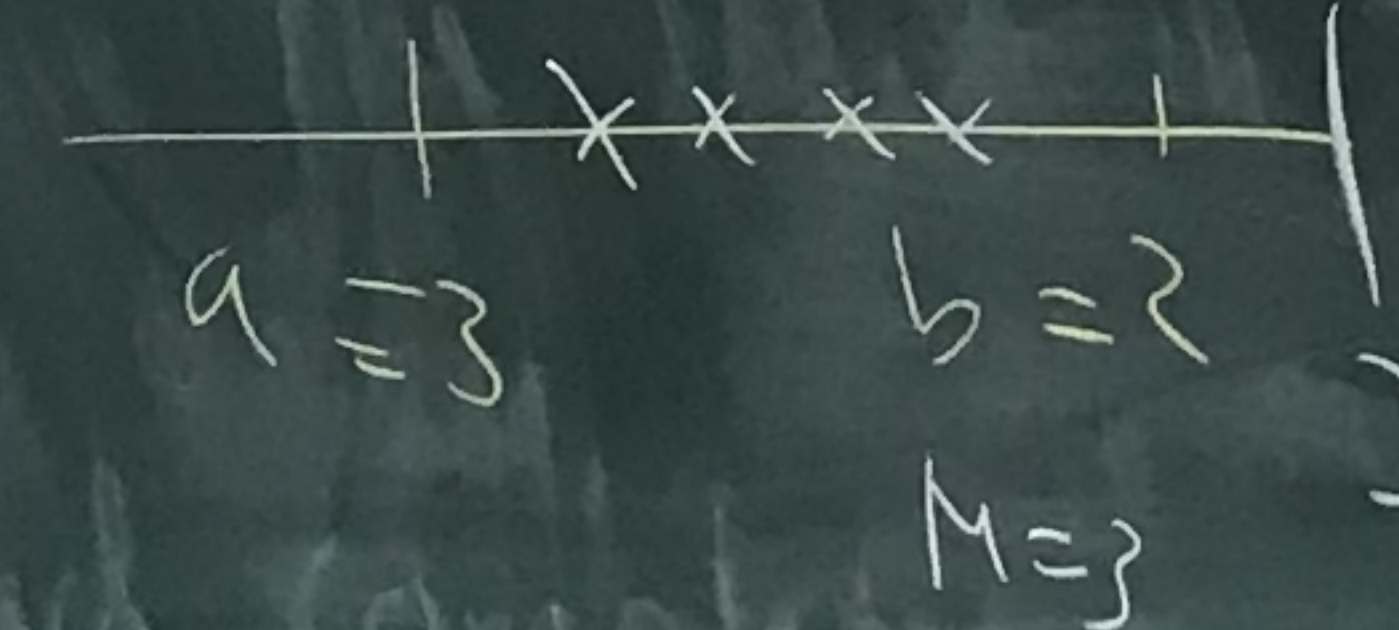
(*) $A \subseteq \mathbb{R}$ is bounded $\iff$ there exists $M > 0$ such that $|x| \leq M \forall x \in A$

(*) $\iff$ (*) (*)
 $M > 0$ take $a = -M$
 $b = M$

$$M = \max\{|a|, |b|\}$$



e.g.



(*) (*) $\implies$ (*) (*) gives $a, b \in \mathbb{R} \ a \leq x \leq b \forall x \in A$

(b) $A \subseteq \mathbb{R}$ is
 there exists

Example of a b

$$A = \{1, 2, 3, 4\}$$

$$A = [0, 1]$$

$$A = (3, 4)$$

(b) $A \subseteq \mathbb{R}$ is not bounded for every $M > 0$
there exists $x \in A$ such that $|x| > M$.

Example of a bounded set

$A = \{1, 2, 3, 4\}$ Take $M = 5$ $A \equiv \text{satisfies } (*)$

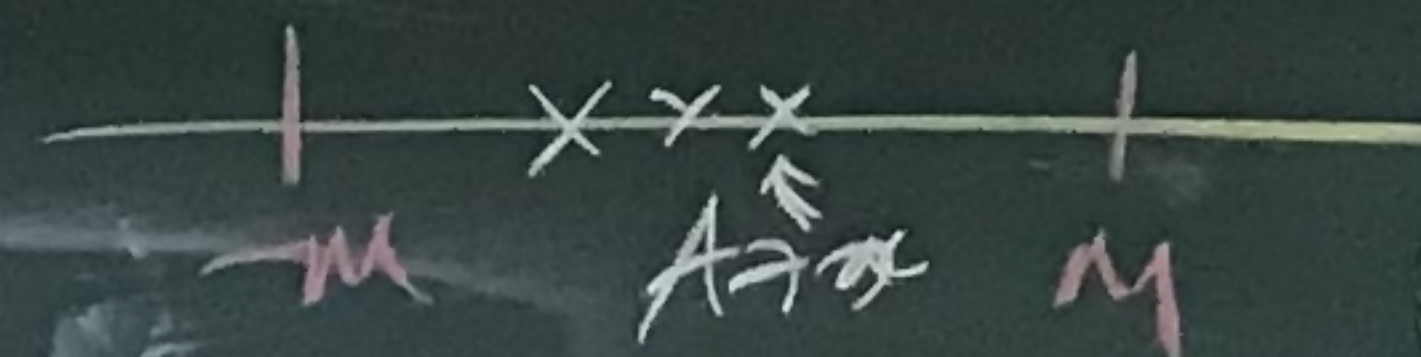
$A = [0, 1]$ Take $M = 2$ $A \equiv \text{satisfies } (*)$

$A = (3, 4)$ Take $M = 4$

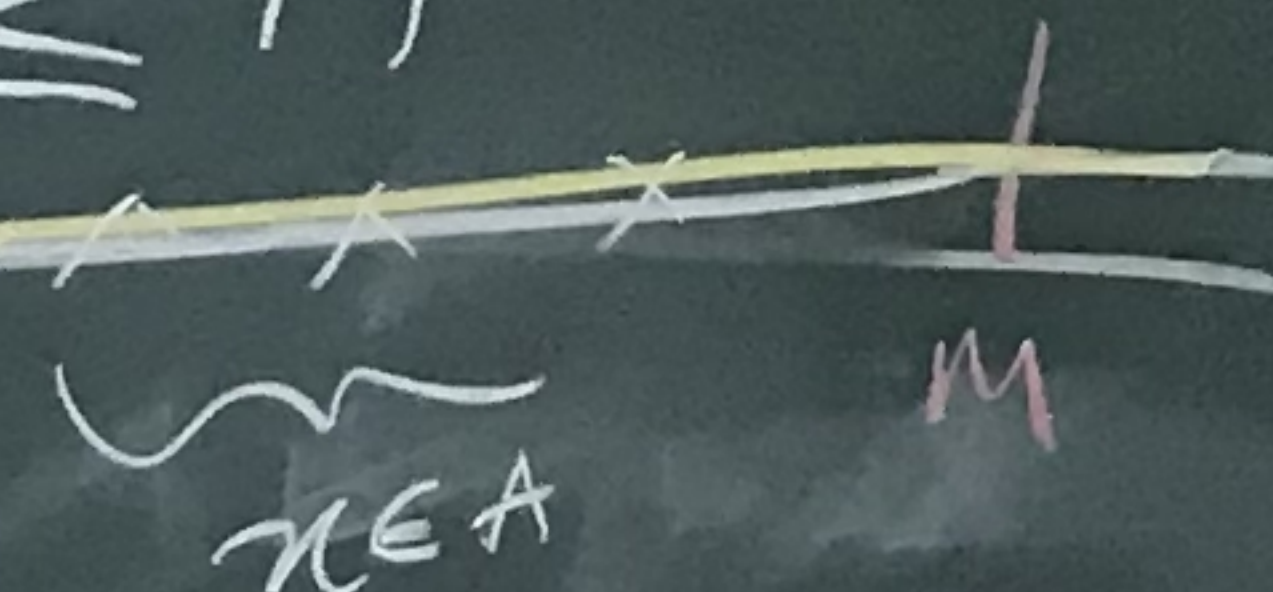
$M > 0$

xxxxx
b=2
M=3

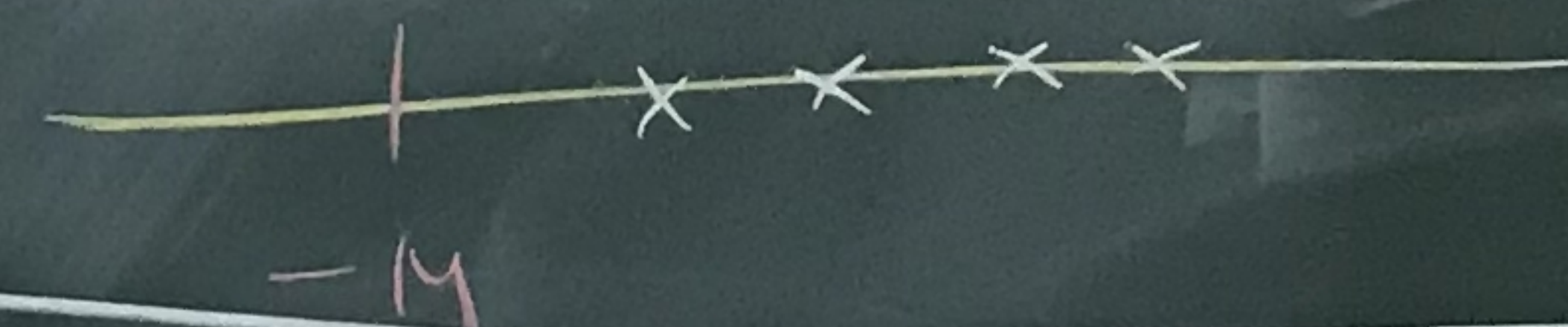
$A \subseteq \mathbb{R}$ Bounded $\iff \exists M > 0 : \forall x \in A \quad |x| \leq M$



$A \subseteq \mathbb{R}$ Bounded above $\iff \exists M > 0 : \forall x \in A \quad x \leq M$



$A \subseteq \mathbb{R}$ Bounded below $\iff \exists M > 0 : \forall x \in A \quad x \geq -M$



Bounded above
 $A = (-\infty, 1)$ $M=2$

Bounded Below
 $A = \mathbb{N}$ $M=0$
 $A = (0, \infty)$
 $A = \emptyset \cup (0, \infty)$

Example: $A = \mathbb{Z}$ is NOT Bounded

Take $M > 0$

Consider $x \in \mathbb{Z}, x = \lfloor M+1 \rfloor + 1$

Observe $|x| \geq M+2 > M$

As $M > 0$ was arbitrary $\implies A$ is NOT bounded.

Ex: A is not bounded below

$$\leq M$$

$$x \in A$$

$$\geq -M$$

Bounded

A is not
bounded
below

bounded.

(c)

Example:

$$A = (3, 4)$$

A is bounded above.

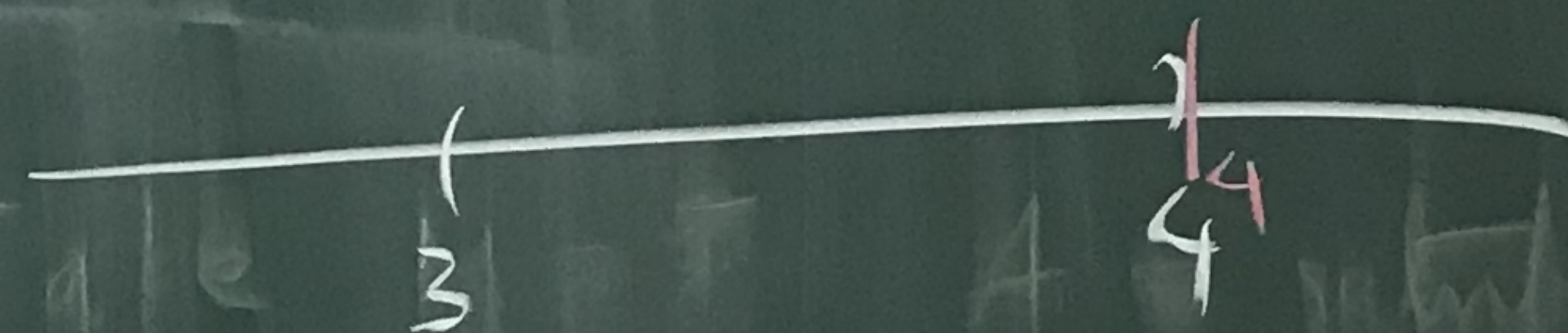
$$M = 4$$

$$M = 5, M = 6, M = 10, \dots$$

(++)

✓

least upper bound



$$\exists \epsilon M = 4 + 0.1 = 3.9$$

$$3.95 \in (3, 4)$$

$$3.95 > 3.9 = M$$

$$4 = \sup(A) = \text{lub}(A)$$

- Idea: - $M > 0$ that satisfies (++) as an upper bound of A
- least upper bound is called supremum.
- 4 is the least upper bound.



Definition (infimum) $\emptyset \neq A \subseteq \mathbb{R}$

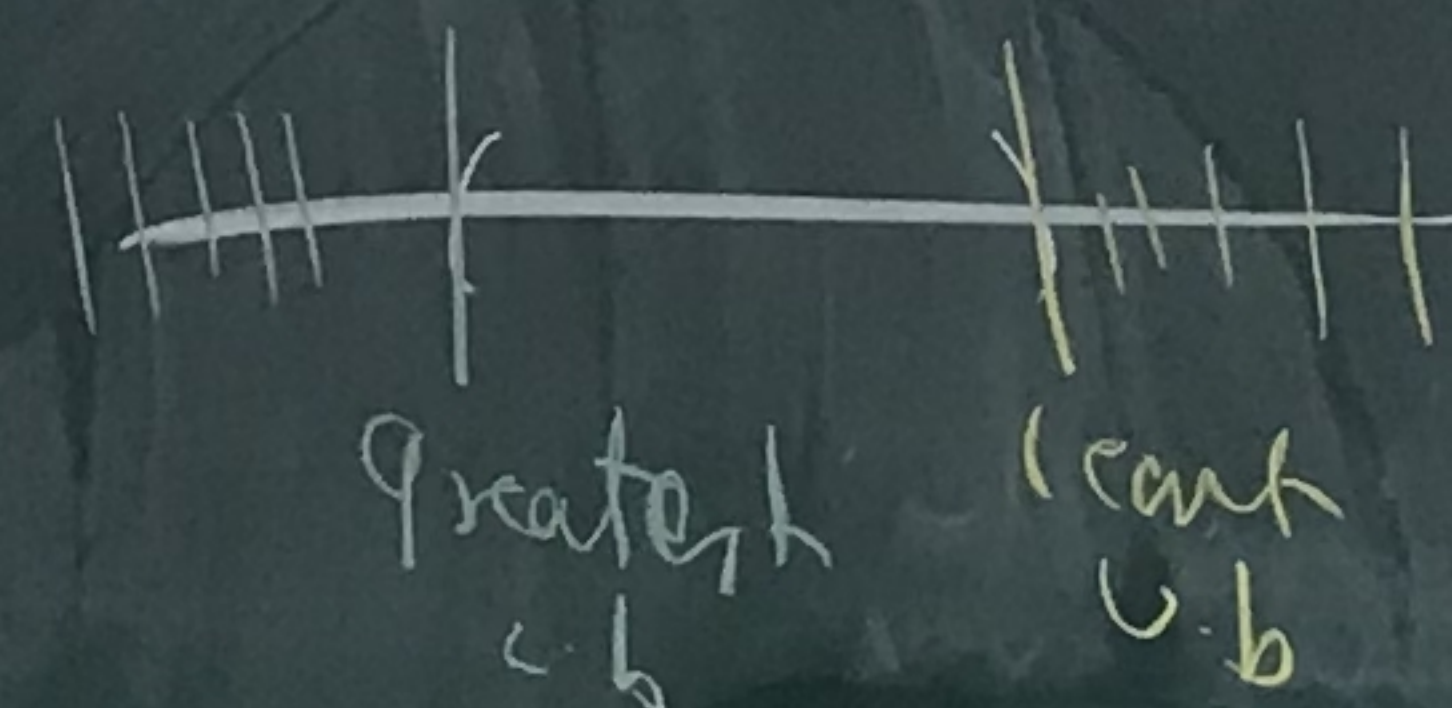
• $M \in \mathbb{R}$ is a lower bound for A

if $x \in A \Rightarrow x \geq M$

• An $\alpha \in \mathbb{R}$ is the greatest lower bound

if - α is a lower bound of A

- if $\beta \in \mathbb{R}, \beta > \alpha$ then β is not a lower bound of A



© Definition -

• $M \in \mathbb{R}$ is a

if $x \in A$

Equivalent
notation

$\alpha = \text{l.u.b.}(A)$

$= \sup(A)$

$= \text{supremum}(A)$

An $\alpha \in \mathbb{R}$ is a bound if

- α is an

(and)

- if $\beta \in \mathbb{R}, \beta < \alpha$ then

© Definition - (Supremum) $\emptyset \neq A \subseteq \mathbb{R}$

$M \in \mathbb{R}$ is an upper bound for A
if $x \in A \Rightarrow x \leq M$.

Equivalent
notation

$$\alpha = \text{l.u.b.}(A)$$

$$= \sup(A)$$

$$= \text{supremum}(A)$$

An $\alpha \in \mathbb{R}$ is the least upper
bound if

- α is an upper bound of A

(and)

- if $\beta \in \mathbb{R}, \beta < \alpha$ then β is NOT an upper bound of A

2 (a)

$$\sup \text{mfm}(B) = 4$$

$$\text{Observe } 4 = \max(B)$$

• First observe $x \in B \Rightarrow x \leq 4$.

So 4 is an upper bound. — (1)

• let $\delta > 0$. Consider $4 - \delta$.

$$\left\{ \begin{array}{l} \text{As } 4 \in B \text{ \& } 4 - \delta < 4 \end{array} \right.$$

So $4 - \delta$ is NOT an upper bound for B.

As $\delta > 0$ was arbitrary we have $\beta < 4 \Rightarrow$ — (2)

β is not an upper bound for B

From (1) and (2), 4 is the least upper bound for B.

$$\text{Infimum}(B) = 2$$

$$\text{Observe } 2 = \min(B)$$

First observe that $x \in B \Rightarrow x \geq 2$

So 2 is a lower bound of B . — (*)

let $\eta > 0$. Consider $2 + \eta$.

As $2 \in B$, $2 + \eta > 2$. So $2 + \eta$ is not a lower bound for B .

As $\eta > 0$ was arbitrary, we have shown

$\beta > 2$ then β is not a lower bound of B . — (**)

(*) & (**) $\Rightarrow$ 2 is the greatest lower bound for B

$$2 \text{ (a)} \quad S = \{ n \in \mathbb{N} \} \cup \left\{ 50 + \frac{1}{n} \mid n \in \mathbb{N} \right\}$$

To show $\inf(S) = 0$

Proof :-

$$n \in \mathbb{N} \Rightarrow \frac{1}{n} > 0$$

$$50 + \frac{1}{n} > 50 > 0$$

0 is a lower bound of S .

Let $\delta > 0$ be given

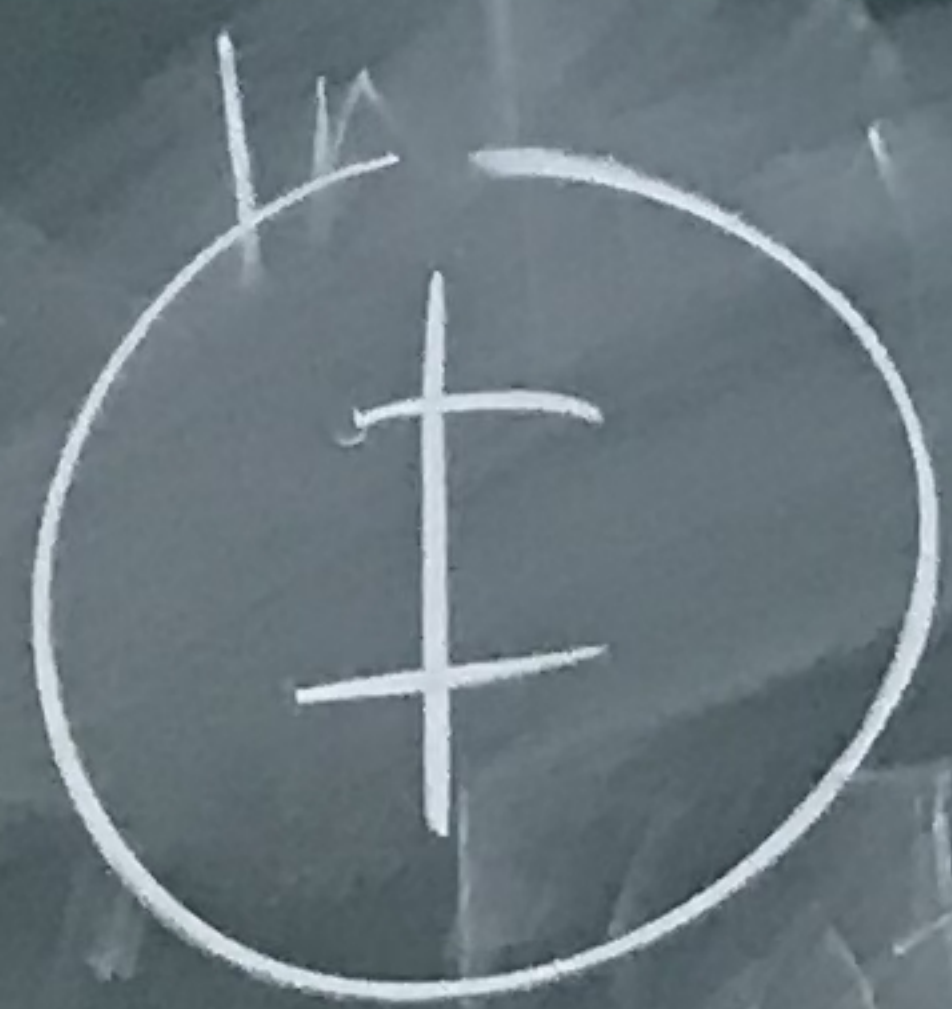
Observe that if

$$n > \frac{1}{\delta} \Leftrightarrow \frac{1}{n} < \delta \quad \text{--- (1)}$$

[Idea - To show δ is not a lower bound]

Take $N = \left\lceil \frac{1}{\delta} \right\rceil + 1$ $N \in \mathbb{N}$ and $N > \frac{1}{\delta}$

0 $\frac{1}{N}$ δ



$\Rightarrow$

$$\frac{1}{N} < \delta$$

But $\frac{1}{N} \in S$ Therefore

$\delta > 0$ is not a lower bound
of S . As $\delta > 0$ was arbitrary

any $\delta > 0$ is not a lower bound of S .

Therefore 0 is the greatest lower bound of S .

4 (a)

$$\inf(A) = u$$

$$\inf(B) = v$$

$$C = \{a+b : a \in A, b \in B\}$$

• What is $\inf(C)$?

pt observation

① - $u = \inf(A) \Rightarrow u$ is a lower bound of $A \Rightarrow a \geq u \forall a \in A$

② - $v = \inf(B) \Rightarrow v$ is a lower bound of $B \Rightarrow b \geq v \forall b \in B$

Take $c \in C$, $c = a+b$ for some $a \in A$ and $b \in B$.

$$(1) \text{ and } (2) \Rightarrow C = a + b \geq u + v = \inf(B)$$

As $c \in C$ was arbitrary $\Rightarrow$ $u + v$ is a lower bound

let $\delta > 0$ and consider $u + v + \delta$

$$\text{observe } u + v + \delta = u + \frac{\delta}{2} + v + \frac{\delta}{2}$$

$$\text{As } \frac{\delta}{2} > 0 \quad \begin{cases} u = \inf(A) \Rightarrow \exists a_0 \in A \quad a_0 < u + \frac{\delta}{2} \\ (3) \quad v = \inf(B) \Rightarrow \exists b_0 \in B \quad b_0 < v + \frac{\delta}{2} \end{cases}$$

$$(3) \Rightarrow a_0 + b_0 < u + \frac{\delta}{2} + v + \frac{\delta}{2} = u + v + \delta$$

$$\exists c_0 = a_0 + b_0 \in C \text{ such that } c_0 < u + v + \delta$$

As $\delta > 0$ was arbitrary $\Rightarrow u + v$ is greatest lower bound

$$a \geq u \quad \forall a \in A$$

$$b \geq v \quad \forall b \in B$$

$$b \in B$$

Problems NOT discussed in class - Worksheet 1

3 - Bartle and Sherbert / Rudin

4 (b) - $\min\{u, v\}$

4 (c) - $\min\{u, 10\}$

5 - induction

6 - ✓

Practice Problems

1 (a), (b), (c)

2. Interesting

(b)

$$\lambda_n = \sqrt{n^2 - n}$$

$-n =$

$$= \frac{-1}{\sqrt{1 - \frac{1}{n}} + 1}$$

Following Results

$$a_n \rightarrow a \text{ \& } b_n \rightarrow b \quad a_n \rightarrow \infty \\ \Rightarrow a_n + b_n \rightarrow a + b$$

$$a_n \rightarrow a, \quad a_n \geq 0, \quad a \geq 0 \\ \Rightarrow \sqrt{a_n} \rightarrow \sqrt{a} \quad a_n \rightarrow a$$

$$a_n \rightarrow a \quad b_n \rightarrow b \quad b_n \neq 0, \quad b \neq 0$$

$$\frac{a_n}{b_n} \rightarrow \frac{a}{b} \quad a_n \rightarrow \infty$$

From definition

Heuristics

$$|x_n - (-\frac{1}{2})| < \varepsilon$$

$\Leftrightarrow$

$$|\frac{-1}{\sqrt{1 - \frac{1}{n}} + 1} + \frac{1}{2}| < \varepsilon$$

$$1 \Rightarrow -\frac{1}{2} - \varepsilon <$$

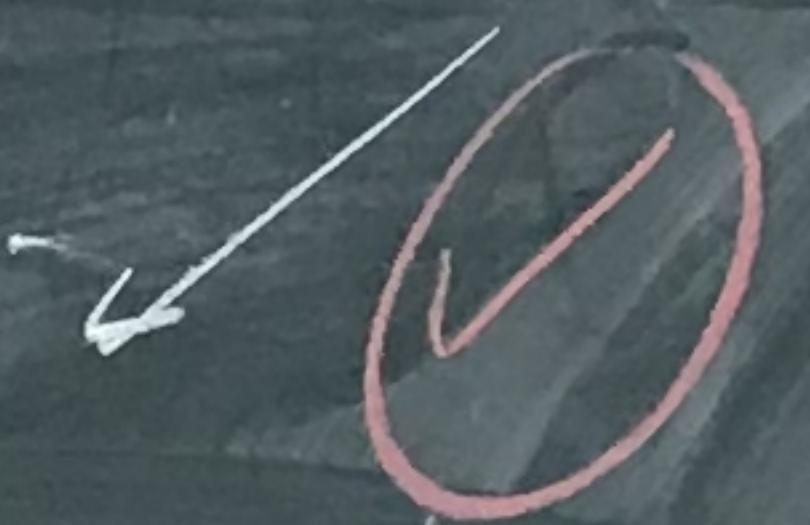
$$\frac{1}{\sqrt{1-\frac{1}{n}}+1}$$

$$< -\frac{1}{2} + \varepsilon$$

$$1 \Rightarrow \frac{1}{2} - \varepsilon <$$

$$\frac{1}{\sqrt{1-\frac{1}{n}}+1}$$

$$< \frac{1}{2} + \varepsilon$$



$$\frac{1}{n} > 0$$

(=)

$$-\frac{1}{n} < 0$$

(=)

$$1 - \frac{1}{n} < 1$$

$$\frac{1}{\sqrt{1-\frac{1}{n}}} > \frac{1}{2} - \varepsilon$$

(=)

$$\sqrt{1-\frac{1}{n}} < 1$$

$$\frac{1}{\sqrt{1-\frac{1}{n}}+1} > \frac{1}{2}$$

(=)

$$\sqrt{1-\frac{1}{n}}+1 < 2$$

(=)

$$\sqrt{1-\frac{1}{n}}+1 < 2$$

(=)

$$-1 + \frac{1}{\frac{1}{2} + \varepsilon} < \sqrt{1-\frac{1}{n}}$$

(=)

$$\left(-1 + \frac{1}{\frac{1}{2} + \varepsilon}\right)^2 < 1 - \frac{1}{n}$$

(=)

$$\frac{1}{n} < 1 - \left(-1 + \frac{1}{\frac{1}{2} + \varepsilon}\right)^2$$

Step 1:

$$0 < a < 1$$

$$\lim_{n \rightarrow \infty} a^n = 0$$

$$a = \frac{1}{b}$$

$$b > 1$$

$$b = 1+h, h > 0$$

$$(1+h)^n = 1 + nh + n(n-1)/2 h^2 + \dots + h^n > nh$$

$$a^n = \frac{1}{b^n}$$

$$= \frac{1}{(1+h)^n}$$

$$= \frac{1}{\sum_{k=0}^n nC_k h^k} < \frac{1}{nh}$$

let $\epsilon > 0$ be given
Observe $|a^n| < \epsilon$

$$\Leftrightarrow a^n < \epsilon$$

$$\text{But } a^n < \frac{1}{nh} \therefore a^n < \epsilon$$

$$\text{if } \frac{1}{nh} < \epsilon \Leftrightarrow n > \frac{1}{\epsilon h} \quad \text{--- } \textcircled{+}$$

$$\text{let } N = \left[\frac{1}{\varepsilon} \right] + 1$$

$$n > N$$

$$\frac{1}{nn} < \varepsilon$$

$$\Rightarrow a^n < \varepsilon$$

$$\Rightarrow |a^n| < \varepsilon$$

As $\varepsilon > 0$ was arbitrary,

$$\forall \varepsilon > 0 \exists N: |a^n| < \varepsilon \quad \forall n > N$$

$$\text{i.e. } \lim_{n \rightarrow \infty} a^n = 0$$

1 (C)

$$x_n = \frac{2^n}{n!}$$

Induction

$$\frac{2^n}{n!} < \frac{9}{2} \left(\frac{2}{3} \right)^n$$

$\forall n \geq 3$

Proof: let $\varepsilon > 0$ be given.

$$0 < \frac{2^n}{n!} < \varepsilon \quad \forall n \geq 1$$

$$\text{So } -\varepsilon < \frac{2^n}{n!} < \varepsilon \quad \forall n \geq 1 \quad (1)$$

$$\forall n \geq 3 \quad \frac{2^n}{n!} < \frac{9}{2} \left(\frac{2}{3} \right)^n \quad (2)$$

Fact: $\lim_{n \rightarrow \infty} \left(\frac{2}{3}\right)^n = 0$

$\left(\frac{2}{3}\right)^n$
 $\forall n \geq 3$

iven.

≥ 1 - (1)

$\left(\frac{2}{3}\right)^n$ - (2)

Let $\varepsilon_1 = \frac{2}{9} \varepsilon$ From fact

$\exists N > 0$ such that $n \geq N$

$\left(\frac{2}{3}\right)^n < \varepsilon_1 = \frac{2}{9} \varepsilon$

(2) $\Rightarrow \frac{2^n}{n!} < \varepsilon \quad \forall n \geq N$

From (1) and (2) $\left| \frac{2^n}{n!} \right| < \varepsilon$
 $\forall n \geq N$

As $\varepsilon > 0$
was arbitrary

$\lim_{n \rightarrow \infty} \frac{2^n}{n!} = 0$

(d) $\lim_{n \rightarrow \infty} n b^n = 0$

$0 < b < 1$

(d)

$$0 < b < 1 \Rightarrow$$

$$\Rightarrow$$

$$b =$$

$$\frac{1}{1+h}$$

for some $h > 0$

$$nb^n$$

$$= \frac{n}{(1+h)^n}$$

$$=$$

$$\frac{n}{1 + nh + \frac{n(n-1)h^2}{2} + \sum_{k=3}^n n h^k h^{n-k}}$$

$$\forall n \geq 2$$

$$<$$

$$\frac{n}{n(n-1)h^2}$$

$$= \frac{2}{(n-1)h^2}$$

$$= \frac{2}{(n-1)h^2}$$

$$= \textcircled{1}$$

$$\left[\because (1+h)^n > n(n-1)h^2 \right]$$

$$\forall h > 0$$

Let $\varepsilon > 0$ be given.

$$|nb^n - 0| < \varepsilon \Leftrightarrow nb^n < \varepsilon$$

$$\lim_{n \rightarrow \infty} nb^n = 0$$

$$k < n-1$$

$$h > 0$$

$$nb^n < \frac{2}{(n-1)h^2} \quad \forall n \geq 3$$

$$nb^n < \varepsilon \quad \text{whenever} \quad \frac{2}{(n-1)h^2} < \varepsilon$$

$$\text{So } \frac{2}{(n-1)h^2} < \varepsilon \quad \text{if} \quad (n-1)h^2 > \frac{2}{\varepsilon}$$

$$\text{Take } N = \max\left\{1 + \frac{2}{\varepsilon h^2}, 3\right\} + 1 \quad \text{or} \quad n > 1 + \frac{2}{\varepsilon h^2}$$

$$n \geq N \quad nb^n < \varepsilon$$

As $\varepsilon > 0$ was arbitrary

$$\lim_{n \rightarrow \infty} nb^n = 0. \quad \square$$