Discrete Derivative as a Tool to Understand Deviations from Exponential Growth

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Abstract

We consider the COVID−19 data of positive test cases in India and use the discrete derivative to detect if the number of positive test cases has stopped growing exponentially. We use a notion similar to moving average of net increase over a symmetric 7-day window. On log-scale, we plot on each day the net increase from three days before that day to three days after that day versus the total number of infections up to that day. If the graph veers of "the straight line" then one can infer that the growth is slower than exponential.

1 Introduction

In this short note, we will try to understand a particular aspect of the trajectory of the COVID−19 infections across the states of India. In India, there have been several intervention measures such as quarantine, lockdown, increase in testing, etc. that are being taken to arrest infection growth. Hence, one would like to understand the effect such measures have had on the infection growth and whether it has stopped growing in an exponential manner. We observe that the relationship between the scaled moving average of the net increases in infections and current infections can be used to infer if the infection growth has deviated from exponential growth.

Understanding infectious disease spread during an epidemic using mathematical models is a very mature field. One of the earliest works dates back to Bernoulli in 1760, [3], who was studying the spread of smallpox. The fundamental mathematical theory for models that are being used today were laid out in a paper by Kermack and McKendrick in 1927 [6]. We quote verbatim from the article.

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... The disease spread from the affected to the unaffected by contact
infection. Each infected person runs through the course of his
sickness, and finally is removed from the number of those who are
sick, by recovery or by death. The chances of recovery or death vary
from day to day during the course of his illness. The chances that the
affected may convey infection to the unaffected are likewise dependent
upon the stage of the sickness. ...
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Then they proceeded to construct a system of ordinary differential equations which are now known as the SIR models where SIR represents three epidemiological classes for the individuals in the population, with S-Susceptible, I-Infected, and R-Removed. Since then various epidemic models have been built on this basic tenet.

Each epidemic model, will postulate a set of assumptions and assign dynamics given by a set of mathematical equations. Then using mathematical techniques one would make inferences on the infection growth. The models are both stochastic as well as deterministic (See [4] for a survey). For any mathematical model a statistical comparison with available data is essential to validate the assumptions made and the dynamics prescribed.

The Ebola epidemic caused by the Western African Ebola virus caused major loss of life and socioeconomic disruption in Guinea, Liberia and Sierra Leone from 2013-2016. Below is a plot of the timeline of the virus

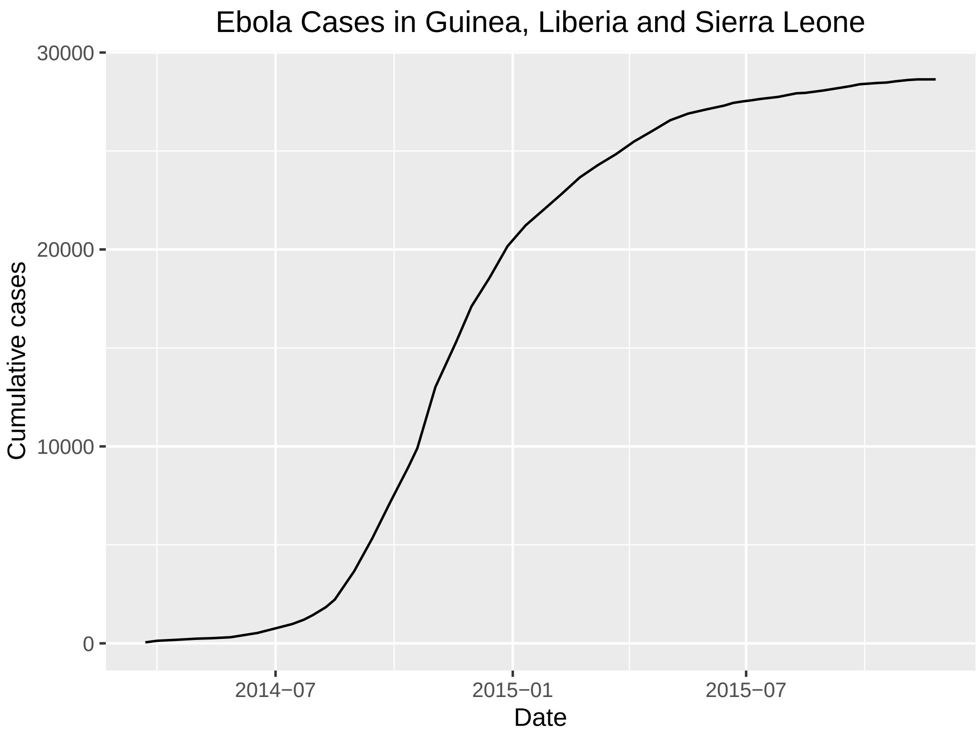

Figure 1: Ebola timeline in Guinea, Liberia and Sierra Leone during 2014-15.

As seen in Figure 1, at the start of the Pandemic, the infection growth appears to be very slow. After which it starts growing quickly around June 2014. Toward the beginning of December 2014, the infection growth starts to slowing down and the curve flattens in late 2015. The phase between June 2014 to December 2014 is referred to as the Exponential phase growth of the infections. The above trajectory is typical of any infection growth during an epidemic. If one is in the midst of an epidemic then one would like to find out if the exponential growth phase has concluded or not. Understanding trajectories of infectious diseases from empirical data is a very mature field.

On our accompanying portal COVID−19 India-Timeline an understanding across States and Union Territories, we use a notion similar to moving average on increase of infections to observe whether the infection growth across States and Union Territories in India is in the exponential phase or not. We were inspired in part by the COVID Trends Website by Aatish Bhatia and Minute Physics.

Remark 1. A word of caution before we begin. The data we use is from the Union Ministry of Health and Family welfare of Government of India website. The data is provided, on a daily basis, as counts of Infected, Recovered and Deceased across the states of India. The number of Infected is the number of positive test results in each state on that day. So we are equating the number of those tested positives as number of infected individuals. This may be an error, because every individual in the population has not been tested. Thus for any inference or conclusion on infection growth we must take into consideration the different policy/rates of testing, population density, quarantine measures, and biological aspects of this epidemic.

The rest of the article is organised as follows. In Section 2, we explain the mathematical preliminaries required to introduce our method. We then show how to apply the same to discrete data. In Section 3, we apply the technique to infection data from States and Union Territories of India. We conclude the note with some remarks about the effects of various interventions on some states.

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2 Preliminaries

In this section we will explain the basic mathematical preliminaries. Section 2.1 and Section 2.2 are intended for the reader who is not familiar with Differential equations, Exponential functions and Logistic functions. In Section 2.3 we discuss how to apply the methods to discrete data.

In most epidemic models, the graph of the total infected over time will closely resemble the following curve:

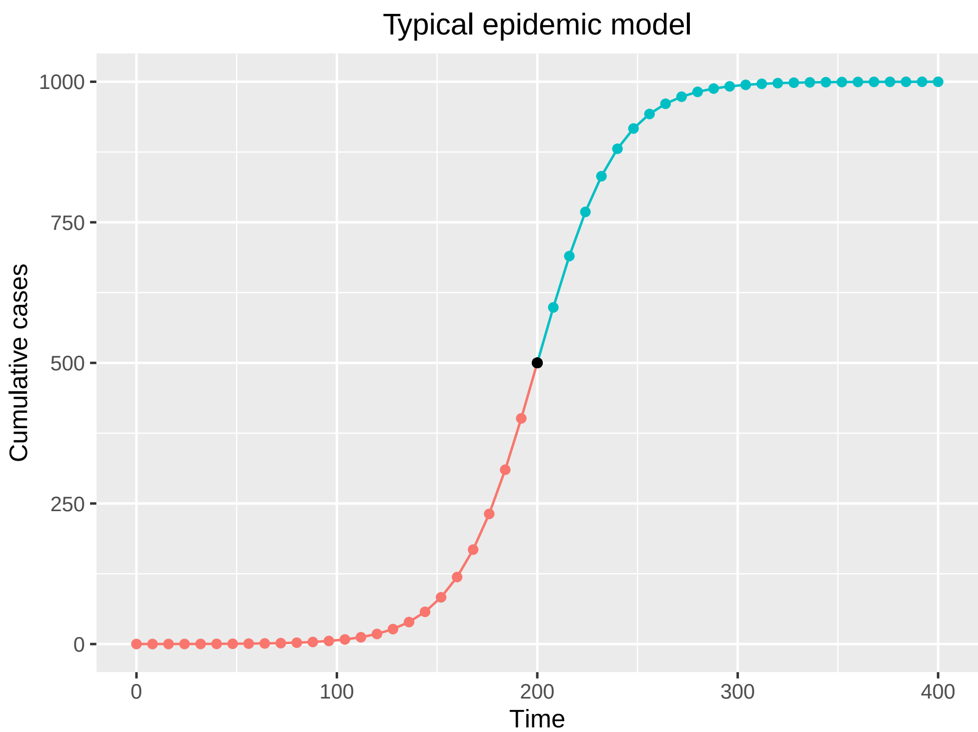

Figure 2: The points on the curve are the values of the logistic function, (defined in Section 2.2), at discrete time points. The graph is formed by linear interpolation between these values.

The curve in Figure 2 is from the very simplistic epidemic model, referred to as the logistic curve. This will be the graph that we work with and use as a rough approximation for the graph of total infected. We shall discuss the logistic curve in detail in Section 2.2. For more general/realistic models see [4] and [5]. In these models also, the graph of total infected vs time has similar broad features as the logistic curve.

Observe that the initial part of the red portion of the graph is growing very quickly and so one can use an exponential function to approximate it. Thus during this red phase, the infections will follow exponential growth. After the point marked black (known as the inflection point) the curve starts to "flatten" out as the number of new infections reduces until it stops. Note that the initial part of the blue curve resembles closely the end of the red curve. Hence, it is hard to identify the inflection point simply by observing the graph of total infections over time.

If we consider the COVID−19 cases, everyone is interested in seeing when the infection exits the exponential growth phase (i.e. the time at which the graph of total infected cases resembles the blue curve). Thus identifying the time where the inflection point occurs is an interesting mathematical question.

2.1 Exponential Function

We will now try to understand the beginning of the red part of the logistic curve using the exponential function. Let $N(t)$ represent the total number of infections at time t. If $N(t)$ is in the exponential growth phase, then we will assume it is given by

$$N(t) = ce^{\alpha t}, \qquad \text{for } t \geq 0. \tag{2.1}$$

Observe that $N(0) = ce^0 = c$. Hence c is the initial count of infected individuals. The parameter α is referred to as the rate of growth or growth rate of the exponential growth. Below is a plot of $N(t)$ given by Equation (2.1), with the initial number of cases to be equal to 100 (i.e., $c = 100$) and the growth rate be 0.4 (i.e. $\alpha = 0.4$).

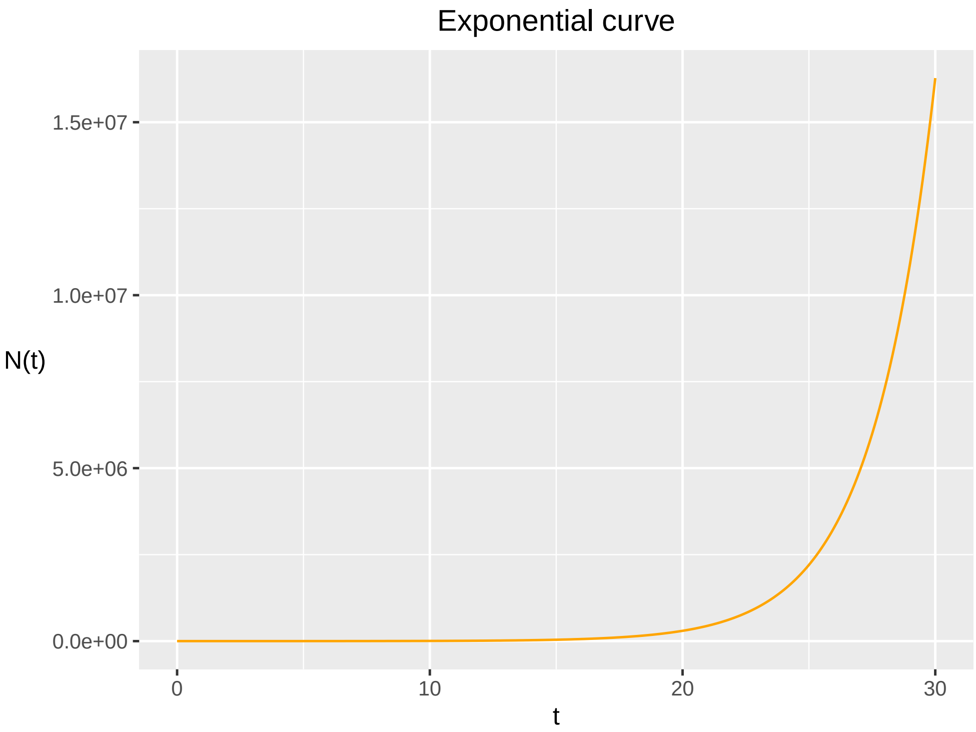

Figure 3: The above figure is an exponential function satisfying the equation: $N(t) = 100e^{0.4t}$

Our objective is to eventually conclude from data if the infection growth is in the exponential phase or not. So we need to identify from the graph of infection growth whether it resembles an exponential function or not. To do this we try to understand the mathematical properties of Equation (2.1). On differentiating both sides of Equation (2.1), we get:

$$\frac{dN}{dt}(t) = \alpha c e^{\alpha t} \qquad \text{for all } t \geq 0.$$

Replacing Equation (2.1) in the, right hand side, above we obtain that:

$$\frac{dN}{dt}(t) = \alpha N(t) \qquad \text{for all } t \geq 0. \tag{2.2}$$

Now, we take log on both sides of Equation 2.2, to obtain

$$\log(\frac{dN}{dt}(t)) = \log(\alpha) + \log(N(t)). \tag{2.3}$$

Thus from Equation (2.2), we see that instantaneous change (or more precisely increase) in cases against the total number of cases will be a straight line with slope α passing through the origin and from Equation (2.3) we see that log of increase in cases against the log of total number of cases will be a straight line with slope 1 and y-intercept $\log(\alpha)$

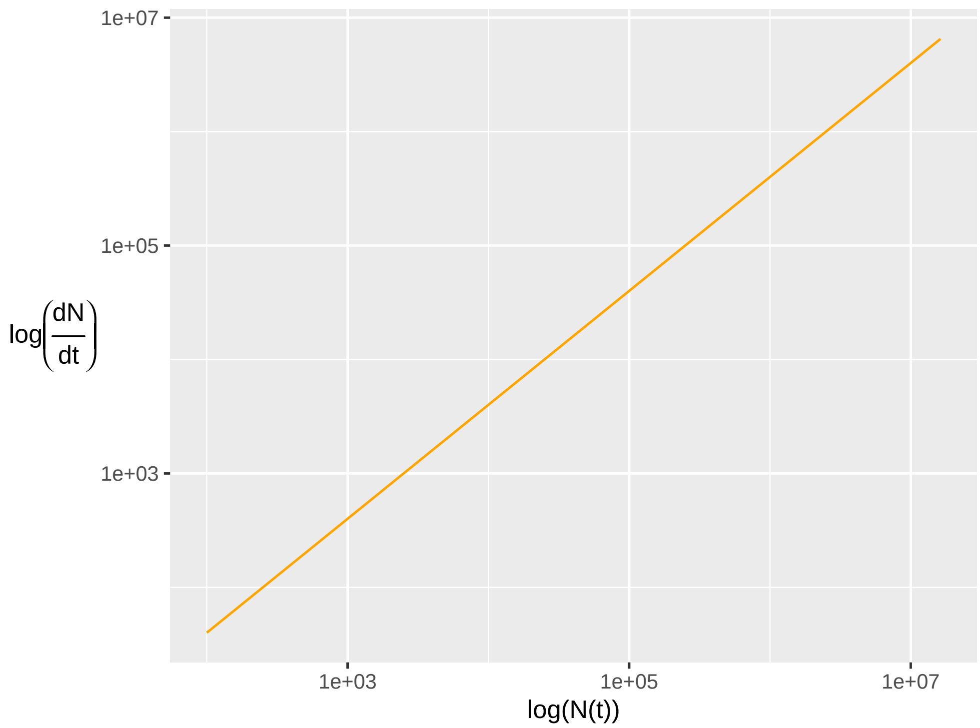

Figure 4: Let $N(t) = 100e^{0.4t}$. The above plot is that of $\log(\frac{dN}{dt})$ over $\log(N(t))$ for exponential growth with the same parameters are Figure 3. The slope of the line is 1 and the intercept is at $\log(0.4) \approx -0.397$

In conclusion, we have seen that the exponential function can be identified by Equations (2.1), (2.2), or (2.3). Exponential growth is often seen in reproduction of bacteria when the growth is unconstrained or in locusts invading a wheat field. An unchecked viral infection, like the COVID−19, may also show exponential growth in the case of no interventions. However such an exponential growth does not last forever due to constraints in resources. For e.g. the wheat field will eventually be completely swarmed by locusts, thus making food scarce and causing their growth rates to slow down. As mentioned earlier, we shall use the simplistic model given by the logistic curve to understand the total number of infections (See Figure 2). We shall discuss this function in the next section.

2.2 Logistic function

Suppose $N(t)$ is described by the Logistic function, then it is given by the equation

$$N(t) = \frac{KN_0e^{rt}}{K + N_0(e^{rt} - 1)}, \tag{2.4}$$

where $t \geq 0$, the parameter K represents the maximum number of infections possible in the respective population, r represents the growth rate and N_0 represents the initial count of infections.

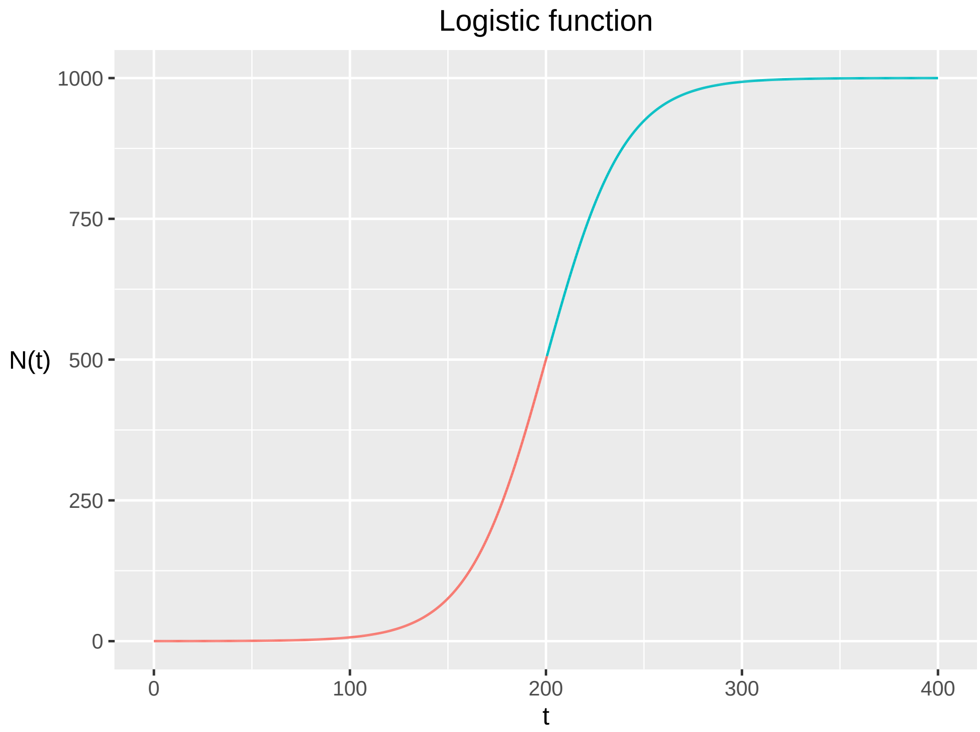

Figure 5: The above curve is a plot of Equation 2.4 with $N_0 = 0.045$, $K = 1000$, and $r = 0.5$.

$K(= 1000$ in Figure 5$)$ is the limiting value of the blue-part of the curve. This can be made rigorous by taking the limit as $t \to \infty$ in Equation (2.4). Now we perform a similar analysis as we did for the exponential function. On differentiating Equation 2.4 with respect to t, using quotient rule followed by chain rule, we have for $t \geq 0$:

$$\frac{dN}{dt}(t) = \frac{(K + N_0(e^{rt} - 1))(KN_0re^{rt}) - (KN_0e^{rt})(rN_0e^{rt})}{(K + N_0(e^{rt} - 1))^2} \tag{2.5}$$

Rearranging the terms and taking rKN_0e^{rt} common in the numerator in the above equation, we have

$$\begin{aligned}
\frac{dN}{dt}(t) &= r(\frac{KN_0e^{rt}}{K + N_0(e^{rt} - 1)})(\frac{(K + N_0(e^{rt} - 1)) - N_0e^{rt}}{K + N_0(e^{rt} - 1)}) \\
&= r(\frac{KN_0e^{rt}}{K + N_0(e^{rt} - 1)})(1 - \frac{N_0e^{rt}}{K + N_0(e^{rt} - 1)}) \\
&= r(\frac{KN_0e^{rt}}{K + N_0(e^{rt} - 1)})(1 - \frac{KN_0e^{rt}}{K(K + N_0(e^{rt} - 1))}) \\
&= rN(t)(1 - \frac{N(t)}{K}) \qquad (2.6)
\end{aligned}$$

where to obtain the second last inequality we have multiplied both the numerator and denominator with K and in the last equality we have used Equation (2.4).

Observe, when N is a very small fraction of K, the ratio $\frac{N}{K}$ very small compared to 1 and we can loosely view $(1 - \frac{N}{K}) \approx 1$. Then in this regime, (2.6) will then loosely resemble

$$\frac{dN}{dt}(t) = rN(t)(1 - \frac{N(t)}{K}) \approx rN(t) \cdot 1 = rN(t)$$

Thus the total population size K acts as the resource and when $N \ll K$, $\frac{dN}{dt}$ is proportional to N. Consequently in this regime the infection is in the Exponential growth (recall (2.2)). Also in this regime, $\frac{dN}{dt}$ versus N will resemble a straight line with slope equal to the rate of growth, r.

Secondly, when N approaches K then we can loosely view $(1 - \frac{N}{K}) \approx 1$ and $\frac{dN}{dt}$ will become 0. The infection growth will come to a halt as 'resources' are not available. This transition can be observed in a sharp manner. For this, as done earlier, we can take log on both sides of (2.6) to obtain

$$\log(\frac{dN}{dt}) = \log(r) + \log(N) + \log(1 - \frac{N}{K}) \tag{2.7}$$

Hence the third term approaches $-\infty$ when N approaches K the transition to the blue part of the curve will be observed in a sharp manner. This is seen clearly in Figure 6.

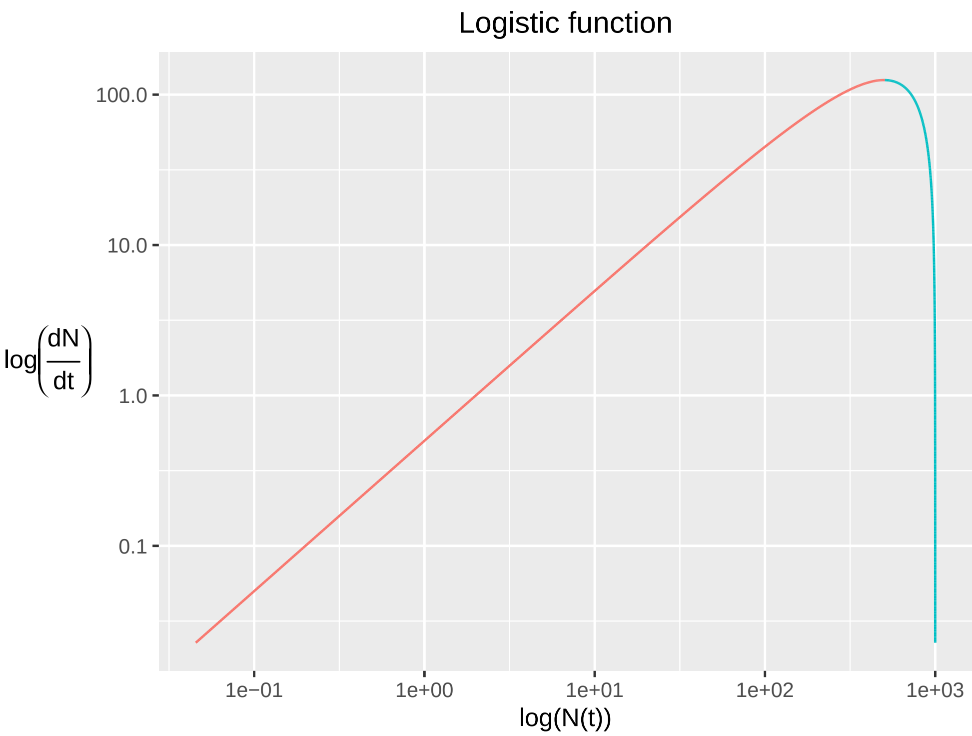

Figure 6: This figure is of the same equation as that of the logistic curve in Figure 5. The above plot is that of $\log(\frac{dN}{dt})$ against $\log(N(t))$.

For discrete data, e.g. daily count of infections of COVID−19 the characterisation given by (2.3) is a key tool to identify rate of growth in the exponential phase and (2.7) can be used to see if it has exited from the exponential phase. To do this from the data one could plot the log of increase in cases against the log of total number of cases and see if the behaviour resembles Figure 4 or Figure 6. We shall discuss this method and its limitations in the next section.

2.3 Discrete Data

In this section we shall see how to apply the techniques developed so far for data that is only available in fixed units of time (e.g. hourly or daily or monthly). Such a data set is referred to as discrete data. First we will evaluate the functions (Exponential from Section 2.1 and respectively Logistic from Section 2.2) at discrete time points $t = 0, 1, 2, \ldots$ and analyse them using only the latter information.

First we take the exponential function given by Equation (2.1) and let the initial number of cases be 100 (i.e., $c = 100$) and the growth rate be 0.4 (i.e. $\alpha = 0.4$). Below is a discrete time plot of $N(t)$ with these parameters:

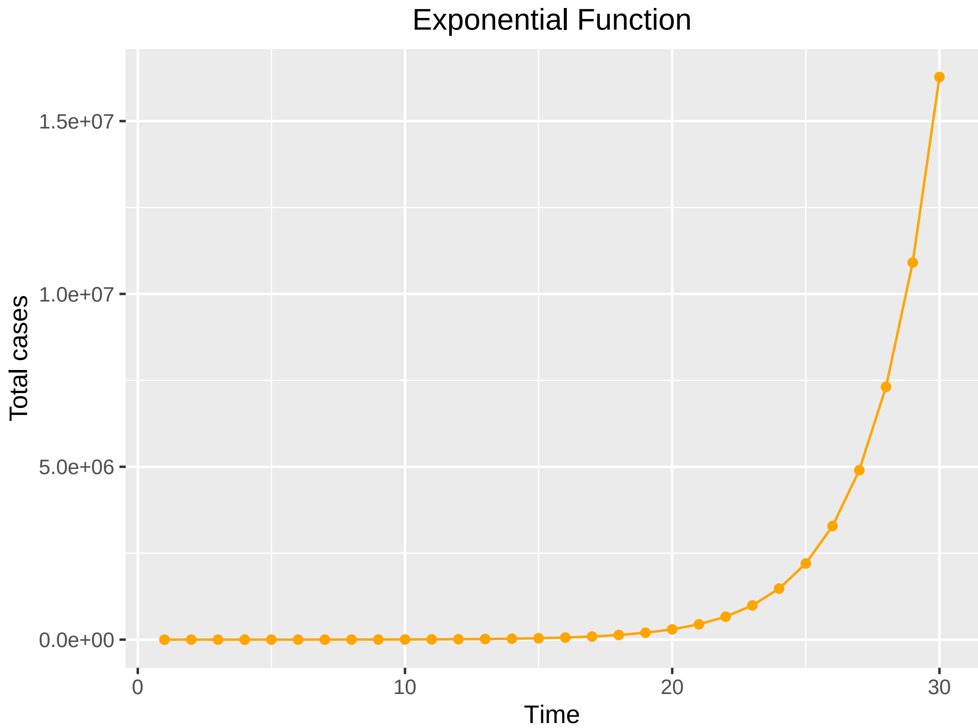

Figure 7: The points above are the values of exponential function $N(t) = 100e^{0.4t}$ at $t = 0, 1, 2, \ldots, 30$ and the curve is formed by linear interpolation between these values.

Now we would like to understand counter parts of Equations (2.2) and (2.3). It is well understood that the derivative of a function represents the instantaneous change in time of the function. Hence if we have information at only discrete time points we will have to make do with the following approximation:

$$\frac{dN}{dt}(t) \approx N(t) - N(t-1). \tag{2.8}$$

Thus we can examine the increase in cases on the previous day versus the total cases up to that day to understand if the data is showing exponential growth. We plot this for the data points from Figure 7.

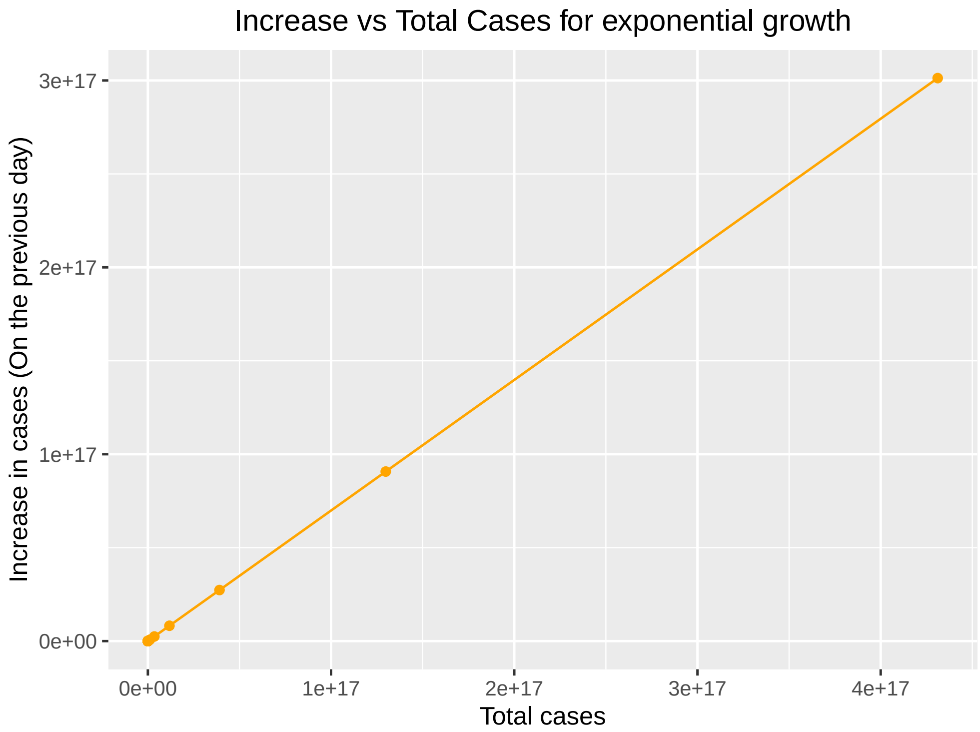

Figure 8: The plot above is of $N(t) - N(t-1)$ on the y-axis versus $N(t)$ on the x-axis.

Observe that in Figure 8, the points near the start overlap are not clearly visible. To make these points visible, one can do the scale change of taking log on both sides, as done to get from Equation (2.2) to Equation (2.3).

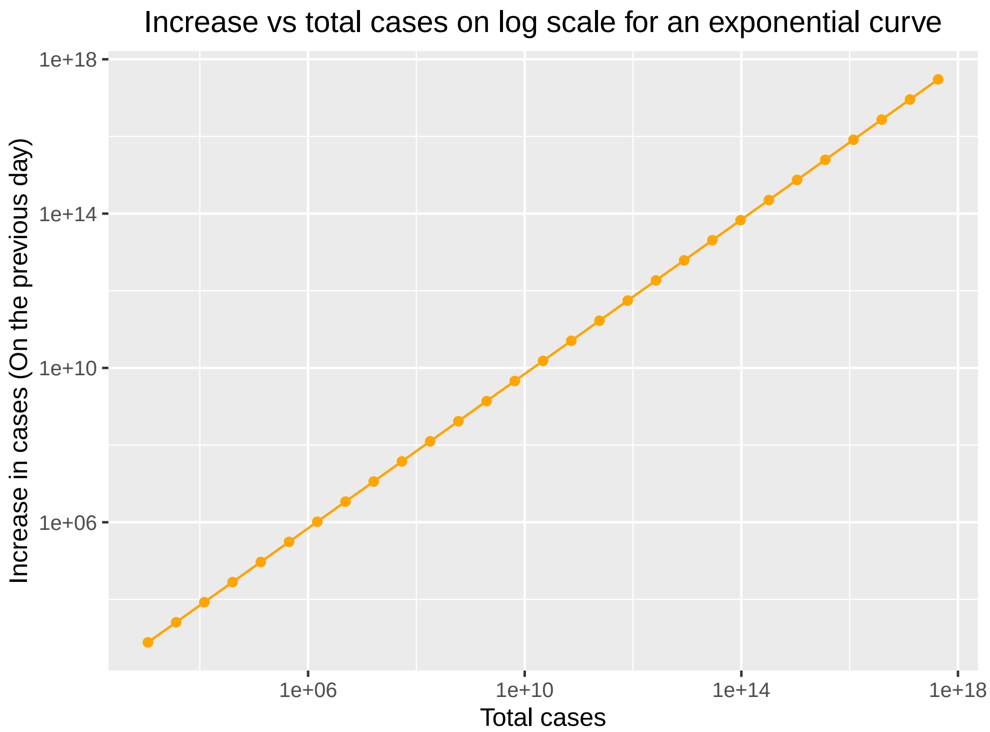

Figure 9: The above graph is the same as figure 8, with both the axes plotted on the log scale.

By converting to log-scale we have made the points equidistant. This is because by plotting both the axes on log scale implies that the numbers $10, 10^2, 10^3, \ldots$ will be equidistant instead of $10, 20, 30, \ldots$ as is the case in the linear scale. We also note that log of increase versus total cases, is a straight line with slope 1 as expected.

Suppose now $N(t)$ is given by the Logistic function represented in Equation (2.4). We evaluate it at the discrete time points $t = 0, 1, 2, \ldots, 30$. The resulting plot would look something like Figure 2. Next we plot the increase (i.e., $N(t) - N(t-1)$) at each of these time points against the total (i.e., $N(t)$). Now if we plot the axes on log scale, one would expect the curve to resemble Figure 6, as seen in the continuous case.

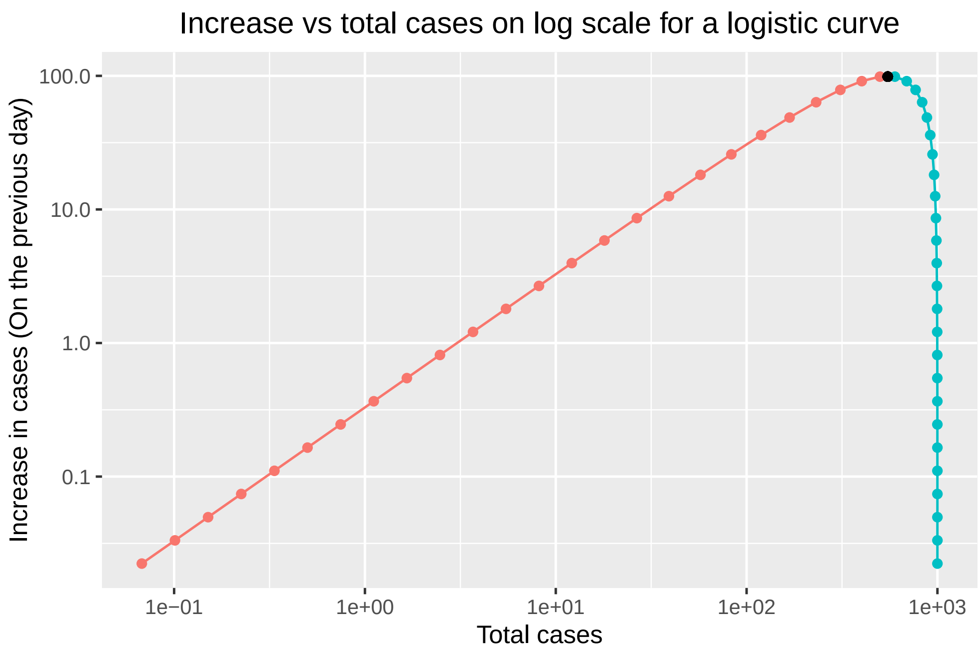

Figure 10: The above graph is the same as that of a typical exponential model (Figure 6), with the increase in cases plotted against the total cases for each discrete time step.

We observe that the red part of the curve in Figure 2 corresponds to the red part of the curve in Figure 10, the inflection point of the logistic curve appears at the peak, and the blue part of the curve that represents the slowing down of the growth is seen as the portion of the above curve that seems to fall sharply off of "the straight line" (i.e. red part). Thus this provides a tool to decide if infection growth has deviated from the exponential phase or not. We shall use it in Section 3 when we study infection times of state and union territories of India.

We conclude this section to understand other slowdowns in infection growth. That is when the infection is growing exponentially but at different rates. Now we consider the case when the rate of exponential growth changes. This is typical in infection growth of COVID−19 for instance. The rate of growth is slowed down to quarantine or lockdowns but still in the exponential phase. For this, we will assume that $N(t)$ is given by

$$N(t) = \begin{cases} 100e^{0.9t} & \text{if } 0 \leq t \leq 30 \\ 100e^{0.9 \cdot 30 + 0.4(t-30)} & \text{if } 30 \leq t \leq 60. \end{cases} \tag{2.9}$$

So we have assumed that initially, the total number of infections is 100. For the first 30 days, the rate of exponential growth is 0.9 (i.e., $\alpha = 0.9$). Then for the following 30 days, the rate is 0.4 (i.e., $\alpha = 0.4$). Suppose we now plot daily increase against the total cases then we obtain:

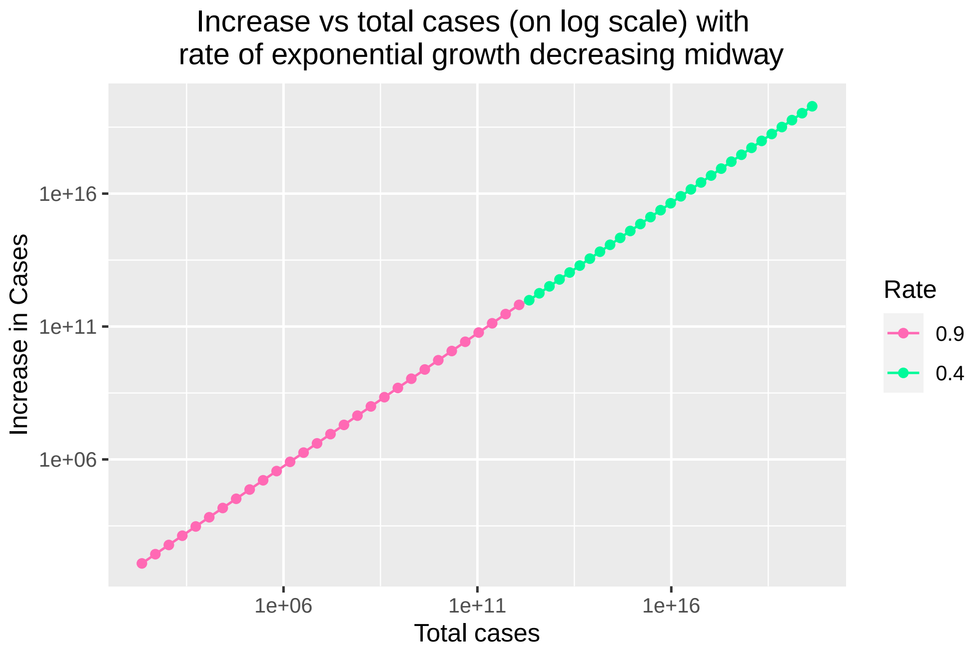

Figure 11: The plot above is that of Equation 2.9 for $t = \{0, 1, 2, ..., 60\}$ plotting on the y-axis $N(t-1) - N(t)$ and on the x-axis $N(t)$ and both the axes plotted on log scale.

It is seen in Figure 11 that regardless of the rate parameter, the graph lies on essentially the same straight line. One can observe from Figure 11 that while the rate parameter is constant, the points remain equidistant. For a higher rate, the points are further apart and for a smaller rate they are closer together.

3 Application: States and Union Territories of India

In this section we shall consider the COVID−19 infection timeline in India. We use the data from the Indian Ministry of Health and Family welfare, where the total counts of the infected individuals (i.e. those who have tested positive for COVID−19), total counts of the individuals who have recovered and total counts of those who have deceased are provided on a daily basis. Figure 12 represents the timeline of the cumulative COVID−19 infections in India.

As we stated earlier in the article, a note of caution. Below we refer to the number of positive test results as the number of infections. This may not be true in most cases as the entire population is not tested. Further, the testing policies vary within states and change with time. Thus the number of positive results depends on the sample of individuals that are being tested. To draw inferences on the number of infected from this data one should take in to consideration this sampling bias. We have not done this. So all graphs drawn below and any inferences concluded are only for the number of reported positive test results.

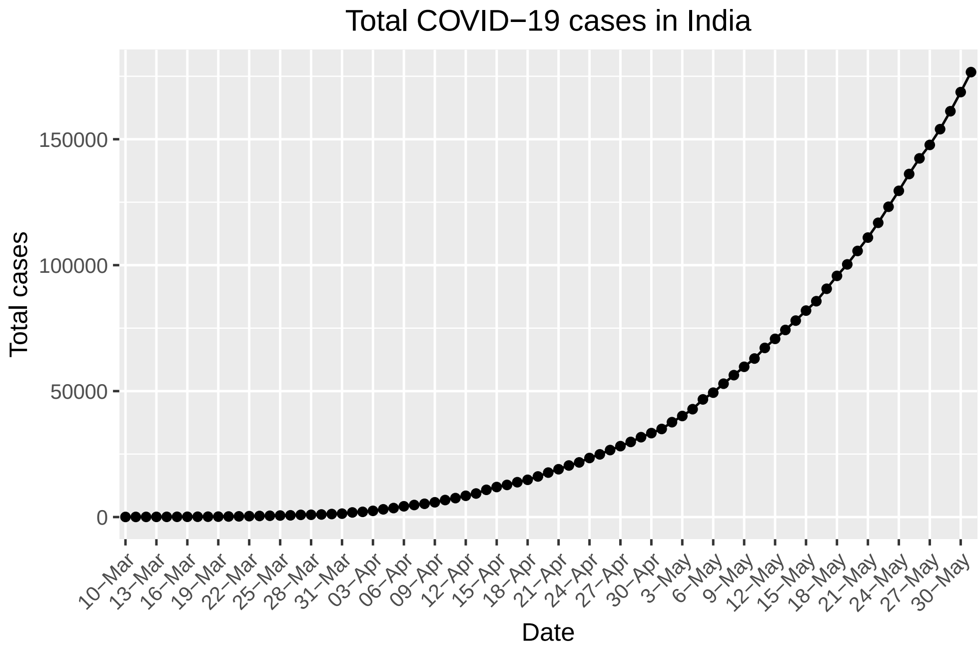

Figure 12: All-India COVID−19 infection timeline.

The above graph seems to resemble an exponential graph as seen in Figure 7. Since the counts for the total Coronavirus cases are updated daily, the data is discrete. We would like to understand whether or not the infection growth is in the exponential phase or not. Further, we would like to understand if the rate of exponential growth is constant, as discussed in Section 2.3. Now, we plot the total cases on the x-axis and the daily increase in cases on the y-axis. Both the axes are on the log scale.

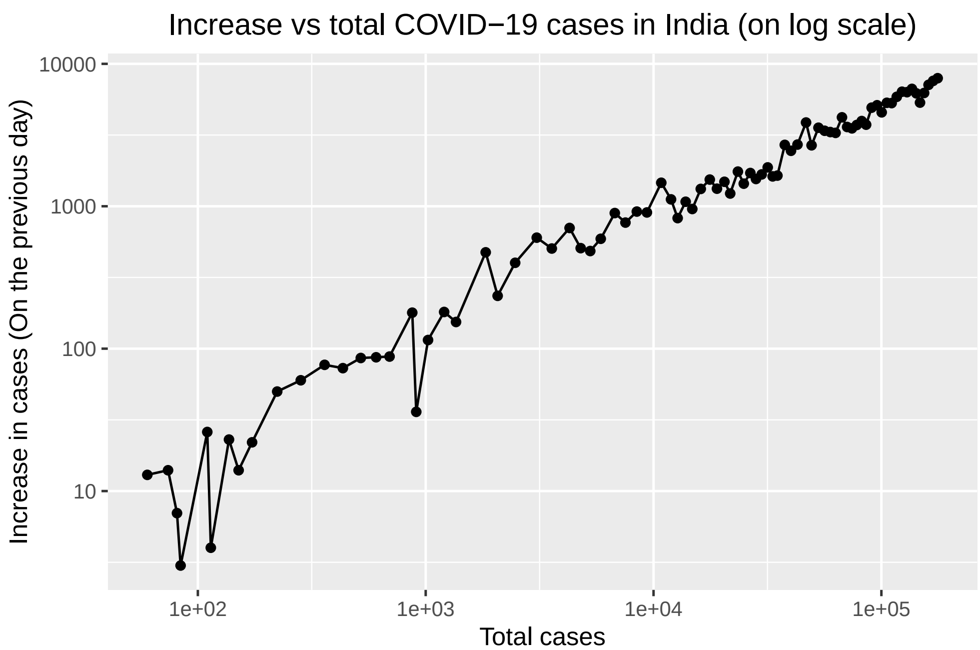

Figure 13: Each point corresponds to a day, the x-axis has the cumulative number of cases till that day. The y-axis has the increase in cases from the previous day.

From Figure 13, we observe that the increase in cases fluctuates a lot. This may be due to clustering together of cases as the testing and reporting is not done continuously. To reduce these fluctuations, we use the notion of the scaled moving average over a symmetric 7-day window. That is, we plot on each day the net increase from three days before that day to three days after that day on the y-axis and the total number of infections up to that day on the x-axis. The y-axis is easily seen as the scaled moving average. More precisely, suppose $N(t)$ was the number of infections on day t then

$$
\begin{aligned}
&\text{Net increase of } N(t) \text{ from three days before day } t \text{ to three days after day } t\\
&\quad = N(t+3) - N(t-3)\\
&\quad = \sum_{k=1}^{3} N(t+k) - N(t+k-1) + \sum_{k=0}^{2} N(t-k) - N(t-k-1)\\
&\quad = 6 \times \text{Moving average of increases over a symmetric 7-day window.}
\end{aligned}
$$

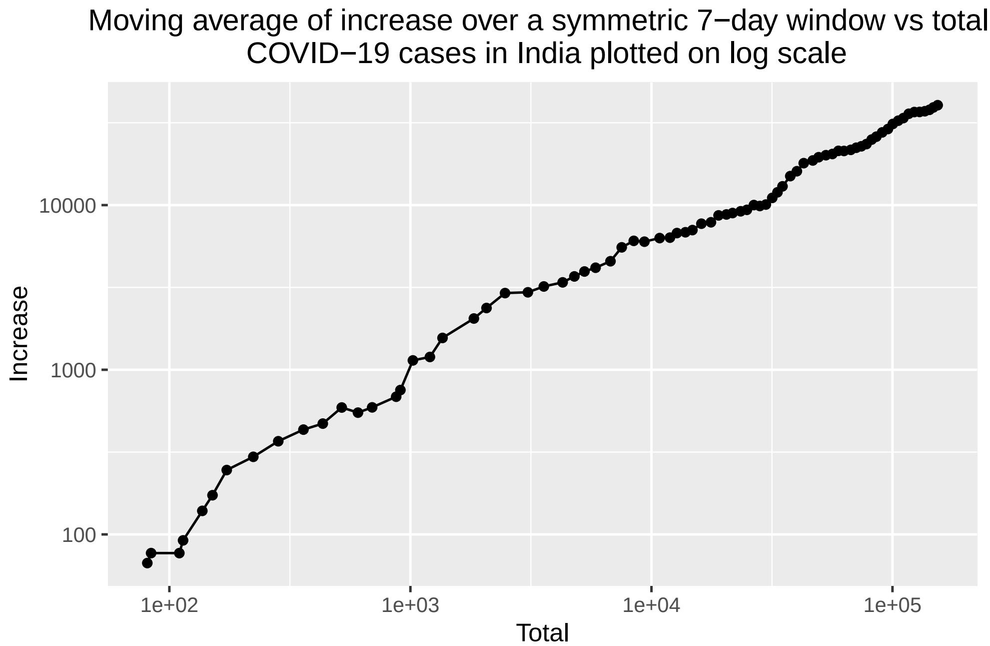

Figure 14: The above graph is the same as the Figure 13 except with the increase taken as the net increase in a 7-day interval around each day.

To re-iterate, for the point that corresponds to 20-April, on the x-axis, we have plotted the increase in cases from 17-April to 23-April on the y-axis, and on the x-axis we have plotted the total number of cases up to 20-April. Both the axes are plotted on the log-scale.

We can conclude that the graph for India, as seen in Figure 14 is quite close to the one seen in Figure 9 as for exponential growth. How it deviates from the perfect exponential growth as seen in Figure 9, is in the aspect that the points aren't equidistant. The graph for India, also doesn't dip below "the straight line" as seen in Figure 10, in the case of logistic growth. It is seen that although this line doesn't deviate from "the straight line" much, the points start getting closer together. Recall from Figure 11 that, when the rate of exponential growth decreases, the points lie closer to one another. Hence, it appears that exponential growth is still being followed although the rate of growth has been decreasing.

For the above trajectory of COVID−19 cases in India, it is seen that until the first $10,000$ cases were reached, the rate was roughly constant and after that it began reducing. During the time the total cases reached 10,000 cases, the infections show a very small growth rate as the points almost overlap each other. Hence the growth rate has been decreasing. This may seem counter intuitive because during early March, the daily increase in cases was much less than what it was during late May. Even though in late May, the daily increase in cases, is the highest it has ever been, the rate of exponential growth is quite low. Despite this, the cases appear to still follow exponential growth. In Figure 14, although none of the axes have time plotted on them, time is represented as the the curve draws itself out.

3.1 Effect of Intervention in India

To curb the spread of the COVID−19 infection, many interventions like lockdowns and social distancing measures are being implemented. A Janta-curfew was conducted on 22-March and on 24-March, when there were 470 active cases in the country, a nationwide lockdown (Phase-1) was announced for 21 days, which has been extended in phases with varying levels of restrictions in Phase-2 (15-Apr to 3-May), Phase-3 (4-May to 17-May) and Phase-4(18-May to 31-May).

To see the effect of the lockdown in various phases on the COVID−19 cases in India, we plot the phases of the lockdown in different colors in the below plot.

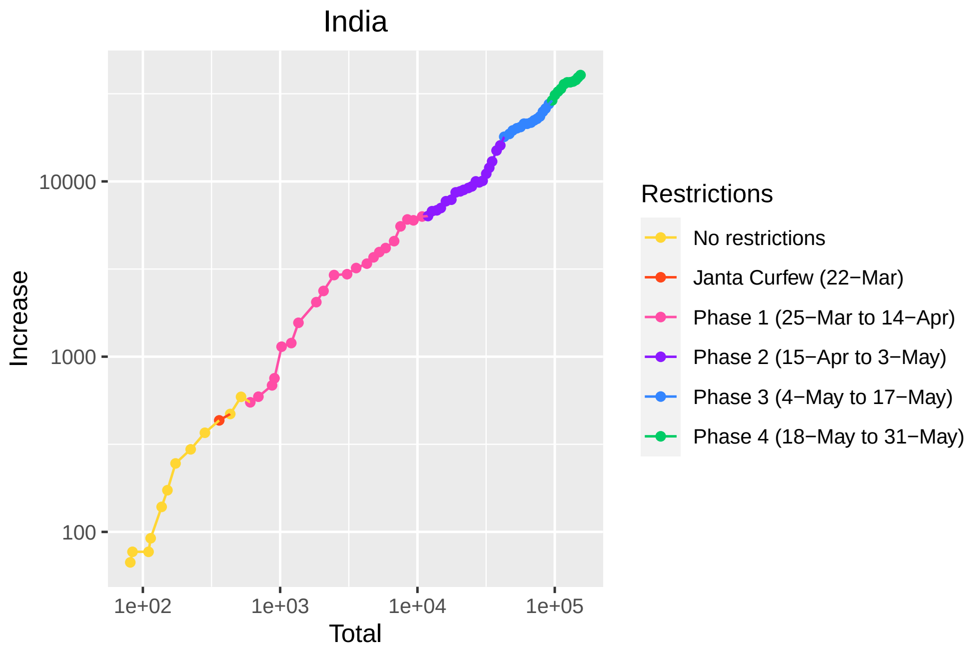

Figure 15: The above graph is the same as that of Figure 14 with the varying restriction phases plotted with different colors. We plot the scaled moving average of increase over a symmetric 7-day window against the total COVID−19 cases in India plotted on log scale, with the Lockdown being represented as different colors of the points.

Consecutive points were seen to be set furthest apart when there were no restrictions. This implies that during that period the growth rate was highest. More recently, although the growth continues to appear exponential, the rate has decreased.

It can be seen that the rate of exponential growth decreased sharply around midway through

Phase-1 of the lockdown. For the COVID−19 infection, there is a significant delay between a patient being infected and symptoms of the infection showing up. This along with some testing and reporting delays, may lead to the effect of the lockdown being seen roughly 1 week - 10 days after the lockdown started. It is observed that while the lockdown was unable to curb exponential growth of cases, it was very successful in significantly reducing the rate of growth.

3.2 Effect of intervention on States and Union Territories

In this section, we look at the trajectories of COVID−19 infection growth in some States and Union Territories in India and the effect of the lockdown on them. The states we consider are Gujarat, Maharashtra, Karnataka, Kerala, Rajasthan and Tamil Nadu. For each of these states, we plot the scaled moving average of the increase in cases over a symmetric 7-day window plotted on the y-axis against the cumulative number of cases on the x-axis.

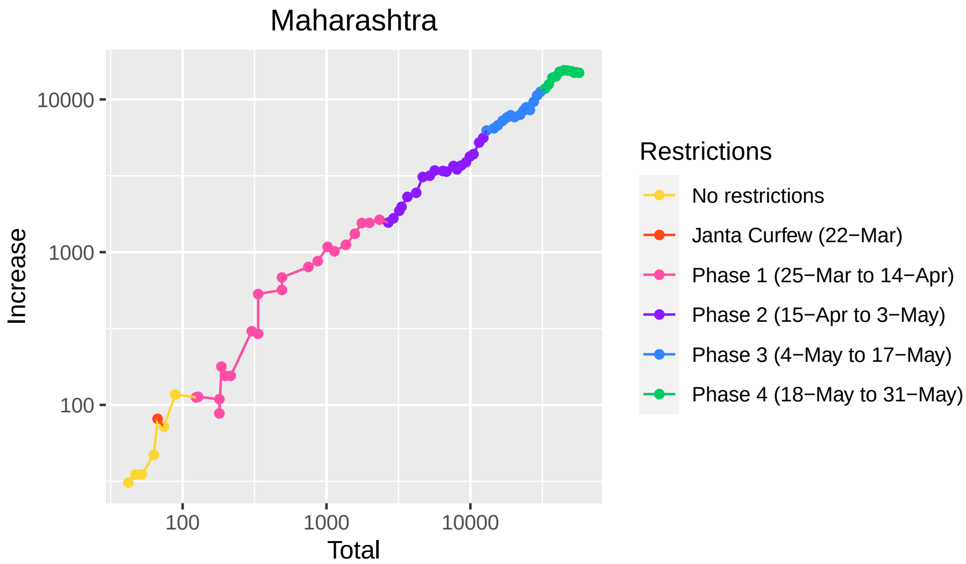

Figure 16: Since the above curve lies approximately on "the straight line," the growth of infections in Maharashtra appears to be in the exponential phase. Much like the curve for India as seen in Figure 15, the points in the curve for Maharashtra start getting closer roughly 7-10 days into Phase-1 of the lockdown. As consecutive points get closer in the Phase 2-4 of the lockdown, the lockdown has effectively reduced the rate of exponential growth.

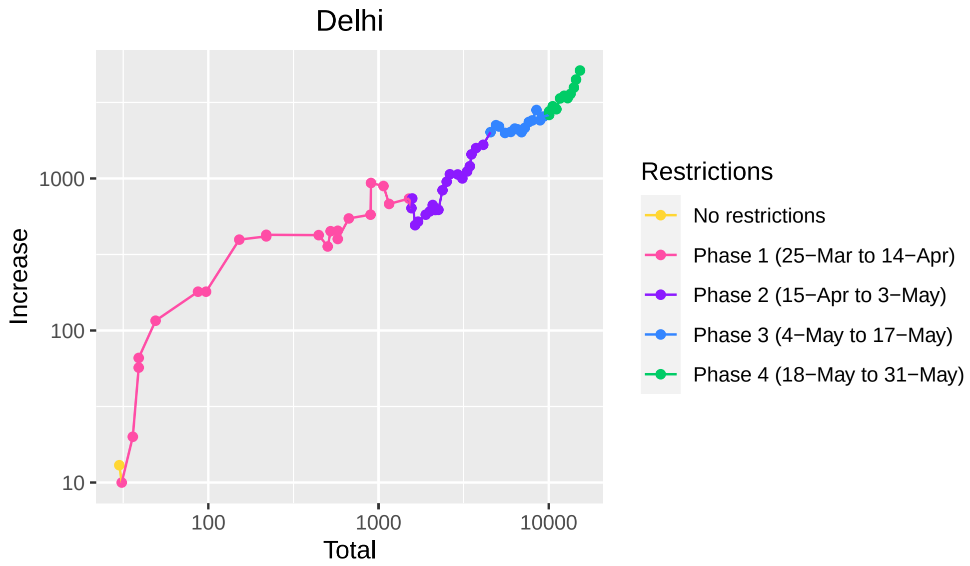

Figure 17: The trajectory followed by Delhi is quite similar to that of Maharashtra (as seen in Figure 16). These graphs differ in the aspect that the growth in the COVID−19 infections in Delhi were at 30 when Phase -1 began and during the intial days of Phase-1 of the lockdown was at a very high rate as the points are apart.

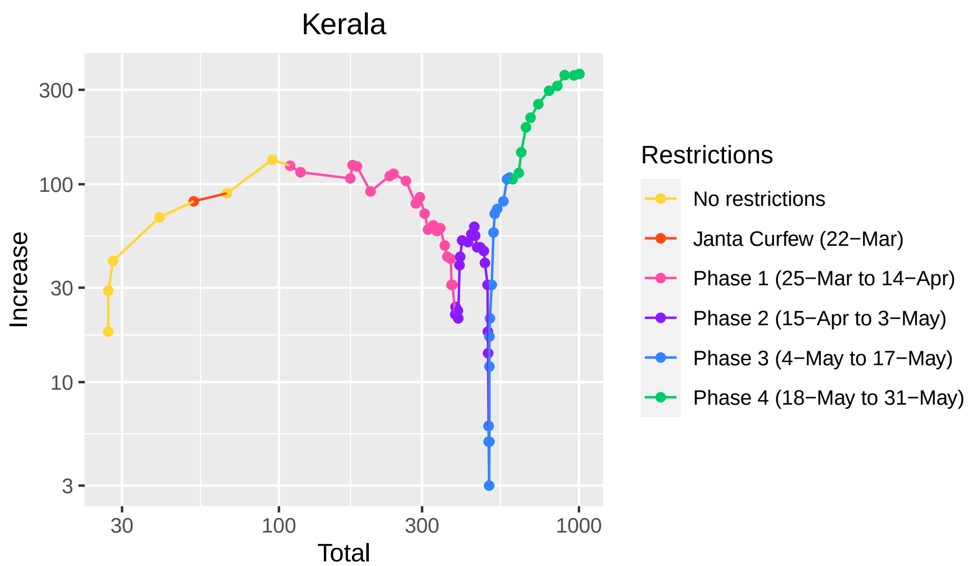

Figure 18: It can be observed that Kerala is not following exponential growth continously over the four phases. Initially, the increase in cases was very large, and after the Phase-1 of the lockdown, a sharp decrease is seen with a clear deviation from exponential growth at the end of Phase-1. The trend continues through Phase-2 although with as slight increase in the cases in the beginning. The cases in Kerala came to a near halt when in the 7-day interval around 6-May (i.e. from 3-May to 9-May) the net increase in cases was 3. There has been a significant increase in the cases during Phase-3 and Phase-4. This is due to the relaxation of the lockdown and can be thought of as the next wave of infections in Kerala, brought about by the migration of individuals both from within India and abroad.

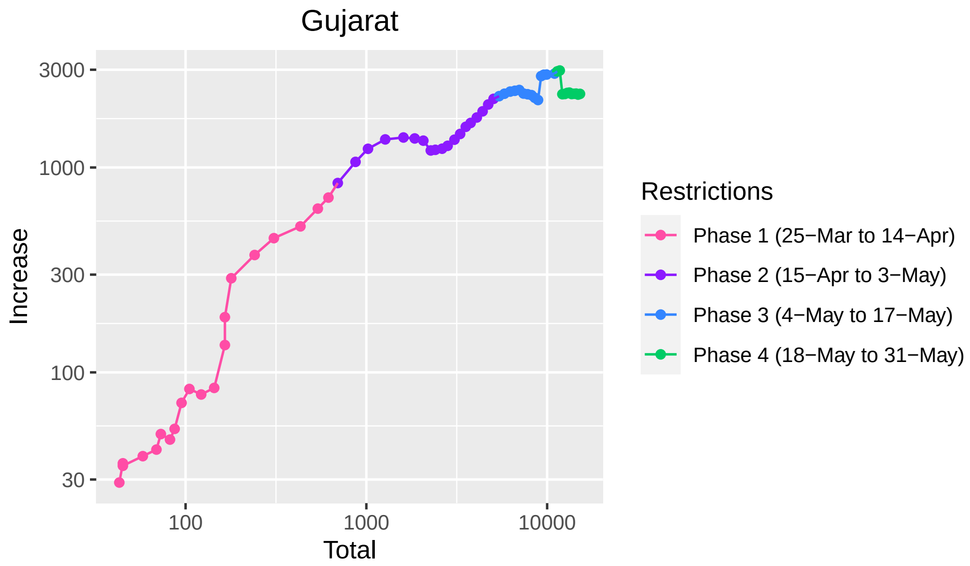

Figure 19: Gujarat seems to be following an infection growth similar to that of Maharashtra and Delhi. The differences are the following. Gujarat had only 37 cases when Phase 1 began and it had a high growth rate during Phase 1. Around 23 April, around midway though Phase-2 of the lockdown, when there were 2400 cases, the growth rate significantly decreases. This may be explained by the different lockdown enforcement policies implemented by the different states.

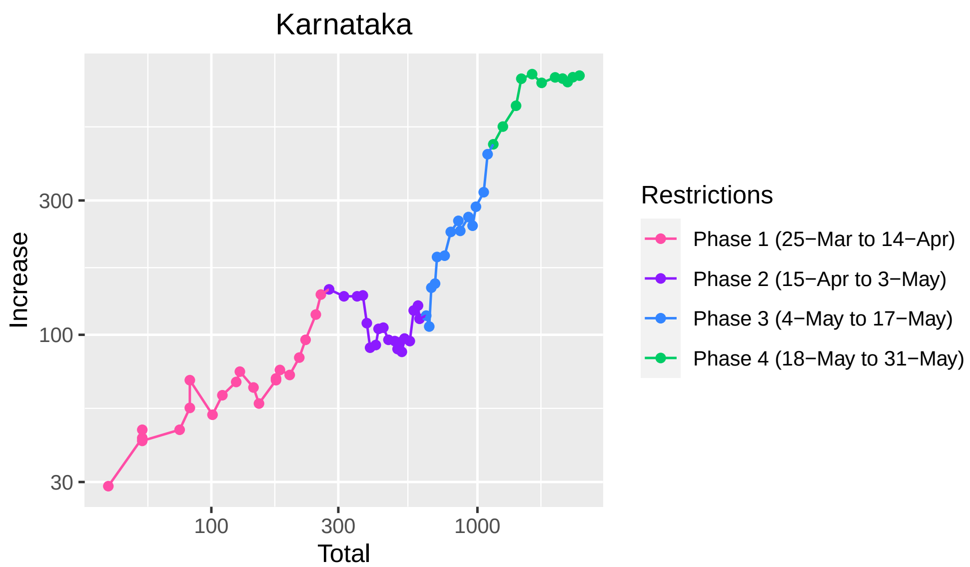

Figure 20: The points get closer to each other during the Phase-1 and we can conclude that the rate of growth reduced due to the lockdown. Phase-2 of the lockdown in Karnataka, saw a significant decrease in the daily counts of cases. The cases appeared to have started falling off "the straight line" for exponential growth. During 20-Apr to 28-Apr in Phase-2 of the lockdown, the increase in cases was seen to be roughly constant, which is a characteristic of non-exponential growth. Phase-3 can be seen to follow exponential growth while having a smaller rate of growth compared to Phase-1 of the lockdown as the points for both these phases in the Figure are roughly equally spaced. Phase-4 of the Lockdown shows exponential growth with a higher growth rate than Phase-3. This may be because Phase-4 saw migration from within India and abroad. It can also be observed that during the second half of Phase-4 the increase in number of cases is roughly constant.

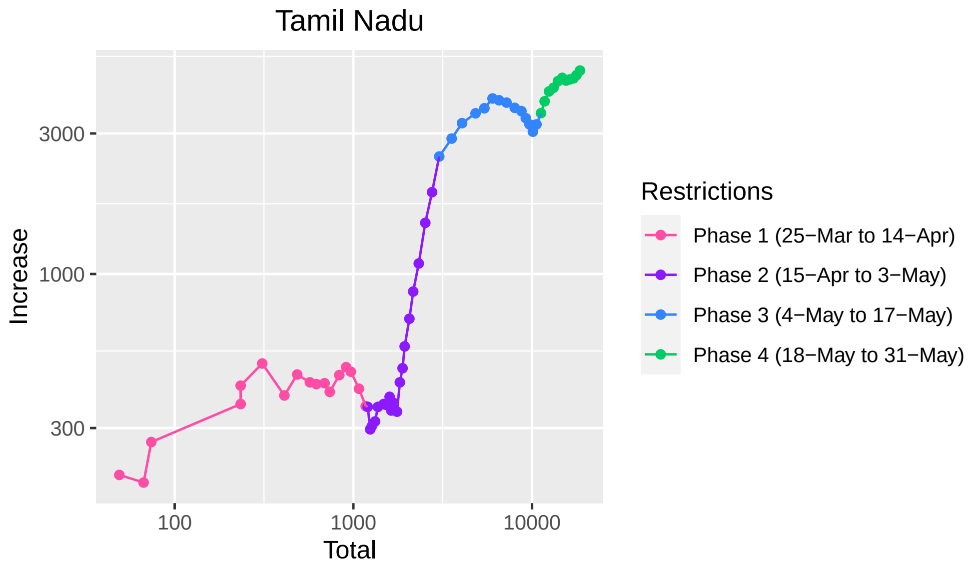

Figure 21: The above plot for Tamil Nadu as plotted follows a different trajectory when compared to many of the states seen previously. The growth rate seems to slow down at the end of Phase-1 of the lockdown when Tamil Nadu reached 1000 cases as the increase in cases seems to stabilize at around 400. Phase-2 of the lockdown in Tamil Nadu records the highest rate of increase of COVID−19 cases. This could be a measure of the testing policy instead of the actual spread of infections since the daily tests done also showed the largest increase during this period as can be seen in the media bulletins. Finally, Phase 3 of the lockdown resulted in the new cases reducing and exponential growth seems to be coming to an end. Phase-4 seems to show exponential growth again with the rate of growth smaller than earlier.

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