Functions of Two Variables

1. Consider $f, g, h, k : \mathbb{R}^2 \to \mathbb{R}$ given by
   (a) $h(x, y) = x + y$
   (b) $f(x, y) = -y^2$
   (c) $g(x, y) = x^2 + y^2$
   (d) $k(x, y) = x^2 - y^2$

   (i) *Graph:* Draw a rough sketch of the graph of f, g, h, and k.
   (ii) *Level Curves:* The level curve or level set at L of a function $u : \mathbb{R}^2 \to \mathbb{R}$ is the $\{(x, y) : u(x, y) = L\}$. For each $L = 0, 9, 100$ draw the level curves of f, g, h, and k.

Derivatives of Functions of Two Variables

Consider $f : \mathbb{R}^2 \to \mathbb{R}$ then the partial derivative w.r.t. x at (x_0, y_0), denote by $f_x(x_0, y_0)$ is given by

$$f_x(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

and he partial derivative w.r.t. x at (x_0, y_0), denote by $f_y(x_0, y_0)$ is given by

$$f_y(x_0, y_0) = \lim_{k\to 0} \frac{f(x_0, y_0 + k) - f(x_0, y_0)}{k}.$$

2. With help of picture explain f_x and f_y as rate of change of f along a specific direction.
3. Let (a, b) be a point in $\mathbb{R}^2$. For each of the functions given in question 1, find the partial derivatives w.r.t x and y at (a, b).
4. For each of the functions given in question 1, identify all (a, b) (if any) where both the partial derivatives w.r.t x and y vanish. What do you think these points characterise ?
5. Can you construct other derivates of $f : \mathbb{R}^2 \to \mathbb{R}$?

Critical Points: local Maxima, local Minima, Saddle Point

6. Let $f, g, h : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = x^2 - 2xy + 6y^2 + 4x - 4y, \; g(x, y) = 1 - x^2 - y^2, \; h(x, y) = \exp(-y^4 - \frac{x^4}{4} + x^2).$$

   (a) Find $f_x(a, b), f_y(a, b)$, $g_x(a, b), g_y(a, b)$, $h_x(a, b)$, and $h_y(a, b)$ for any $(a, b) \in \mathbb{R}^2$.
   (b) Find the Critical points of f, g, h i.e. for each of the functions identify all (a, b) (if any) where both the partial derivatives w.r.t x and y vanish.
   (c) Can you classify them as local Maxima, local Minima or Saddle Point ?
7. Let $a \neq 0, b \in \mathbb{R}, c \in \mathbb{R}$, $f : \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x, y) = ax^2 + bxy + cy^2$.
   (a) Without using partial derivatives, can you identify conditions on a, b, c that will identify all the critical points of f and their identiities as local maxima, local minima or saddle points.
   (b) Recall the second derivative test in one variable. Can you formulate a suitable second derivative test for $f : \mathbb{R}^2 \to \mathbb{R}$?