Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers.

1. Provide examples of sequences $\{a_n\}$ that satisfy each of the statements below.

   (a) For all $\epsilon > 0$, for all but finitely many $n \in \mathbb{N}$

   $$a_n < 5 + \epsilon \text{ and } a_n > -11 - \epsilon$$

   Solution:
   Example 1: $a_n = 4$ for all $n \geq 1$

   Example 2: $a_n = -11 + \frac{(-1)^n}{n}$ for all $n \geq 1$

   Example 3: $a_n = 5 + \frac{1}{n}$ for all $n \geq 1$

   Example 4: Any sequence a_n which has no limit points in $(-\infty, -11) \cup (5, \infty)$ will satisfy the above hypothesis.

   Exercise: Prove that a_n satisfies (a) if and only if $-11 < \liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n < 5$. □

   (b) For all $\epsilon > 0$, there are infinitely many $n \in \mathbb{N}$ such that

   $$| a_n - L | < \epsilon$$

   for $L = -1, 0, 3$ and $a_n \notin \{-1, 0, 3\}$ for all $n \geq 1$
   Solution:
   Example:

   $$a_n = \begin{cases} -1 + \frac{1}{n} & \text{if } n = 3k \text{ for some } k \geq 1 \\ \frac{1}{n} & \text{if } n = 3k+1 \text{ for some } k \geq 1 \\ 3 + \frac{1}{n} & \text{if } n = 3k+2 \text{ for some } k \geq 1 \end{cases}$$

   Exercise: Let S be the set of limit points of the sequence a_n. Prove that a_n satisfies (b) if and only if $S \supset \{-1, 0, 3\}$. □

   (c) For $a > 0$ there are infinitely many $n \in \mathbb{N}$ such that

   $$a_n > a \text{ and}$$

   there are infinitely many $n \in \mathbb{N}$ such that

   $$a_n < -a.$$

   Solution:
   Example : Let $\alpha > 0$ and $a_n = (-1)^n n^\alpha$ for all $n \geq 1$
   Exercise: Prove that a_n satisfies (c) if and only if $-\infty = \liminf_{n\to\infty} a_n < \limsup_{n\to\infty} a_n = \infty$. □

2. Write the below statements using logical notation:

   (a) *For every $\epsilon > 0$ there are infinitely many n such that distance of a_n to 0 is less than ϵ*
   Solution:
   In Logical Notation: $\forall \epsilon > 0$, $\forall N \geq 1$ there exists $m_0 > N$ such that $| a_{m_0} | < \epsilon$
   Note: m_0 depends on ϵ and N □

   (b) *For every $\epsilon > 0$, all but finitely many elements of the sequence a_n are below $11 + \epsilon$ and inifinitely many above $11 - \epsilon$*
   Solution:
   In Logical Notation: $\forall \epsilon > 0$, $\exists N \geq 1$ such that for all $n > N$ such that $a_n < 11 + \epsilon$, $\forall \epsilon > 0$ and $\forall N \geq 1$ there exists $m_0 > N$ such that $a_{m_0} > 11 - \epsilon$
   □

3. Consider the following statements:

   (a) For every $\epsilon > 0$ there exists $N > 0$ such that $| a_n - L | < \epsilon$ for all $n > N$.

   (b) There is a $C > 0$ such that for every $\epsilon > 0$ there exists $N > 0$ such that $| a_n - L | \leq C\epsilon$ for all $n \geq N$.

   (c) For every $N > 0$ there exists $\epsilon > 0$ such that for all $n > N$ implies $| a_n - L | < \epsilon$.

   (d) There exists $N > 0$ such that for all $\epsilon > 0$ and $n > N$ implies $| a_n - L | < \epsilon$.

   (e) For every $\epsilon > 0$ and for all $n \geq 1$, there exists $N > 0$ such that $m > N$ implies $| a_m - L | < \epsilon$.

   (f) For every $\epsilon > 0$ and for all $n \geq 1$, there exists $N > 0$ such that $N > n$ and $| a_N - L | < \epsilon$.

   Decide which of the above versions are equivalent to the definition of

   $$\lim_{x\to\infty} a_n = L$$

   and which are not. For those that are not equivalent to $\lim_{n\to\infty} a_n = L$ determine, in as simple a language as possible, what they really define. Find examples (if they exist) of sequences that satisfy the definition and of sequences that don't satisfy it.

ANSWER: $(a) \iff (b) \iff (e) \iff \lim_{x\to\infty} a_n = L$, $(d) \Longrightarrow \lim_{m\to\infty} a_n = L \Longrightarrow (c), (f)$.

Exercise: Show that if a sequence satisfies (d) if and only if $\exists N > 0$ such that $a_n = L$ for all $n > N$. Show that if a sequence satisfies (c) or (f) then L is a limit point of a_n but need not be its limit.