Writing of Mathematics **Quiz 1** **Semester I 2018/19**

Your name: **Solution** September 27th, 2018

1. Let $n \geq 1$, $x_n = \frac{n^3}{(1+p)^n}$ with $p > 0$. Decide if the sequence converges or not.

Solution: For any $n \geq 1$, by the Binomial expansion[4]

$$(1+p)^n = \sum_{k=0}^{n} \binom{n}{k} p^k.$$

All terms in the above expansion are positive numbers. So for $n \geq 6$ and it is easily seen from the above that

$$(1+p)^n \geq \binom{n}{4} p^4 = \frac{n(n-1)(n-2)(n-3)}{24} p^4 \geq \frac{n^4 p^4}{24}(1-\frac{1}{n})(1-\frac{2}{n})(1-\frac{3}{n}) > \frac{n^4 p^4}{24} \cdot \frac{1}{2^3}.$$

We have used the fact that for all $n \geq 6$, $1 - \frac{1}{n}$, $1 - \frac{2}{n}$, and $1 - \frac{3}{n}$ are all greater than $\frac{1}{2}$. So, for all $n \geq 6$,

$$(1+p)^n < n^4 \frac{p^4}{192}.$$

Therefore for all $n \geq 6$,

$$x_n = \frac{n^3}{(1+p)^n} < n^3 \cdot \frac{192}{n^4 p^4} = \frac{192}{p^4} \cdot \frac{1}{n}.$$

Let $\epsilon > 0$ be given. Let $N > \frac{192}{p^4 \epsilon}$. So for all $n \geq N$, we have

$$\frac{192}{p^4} \cdot \frac{1}{n} < \epsilon.$$

Let $N_1 = \max\{N, 6\}$. For all $n \geq N_1$ we have

$$0 < x_n < \frac{192}{p^4} \cdot \frac{1}{n} < \epsilon.$$

This in turn implies that for all $n \geq N_1$,

$$| x_n | < \epsilon.$$

As $\epsilon > 0$ was arbitrary we have shown that

$$\lim_{n \to \infty} x_n = 0.$$

In Homework you were asked ot consider $x_n = \frac{n^\alpha}{(1+p)^n}$ with $\alpha, p > 0$. Can you amend this proof to finish that case ?

[4] A proof of the formula can be done by the method of induction.