Your name: **Solution** November 1st, 2018

1. Classify all groups of order 325 and 26.

Solution:

Case 1: $335 = 5^2 \times 13$

Let n be the number of Sylow 5 subgroups. Then

$$n \equiv 1 \mod 5 \text{ and } n \text{ divides } 13.$$

This implies $n = 1$. So there is one subgroup H of order 25 in G.

Let m be the number of Sylow 13 subgroups. Then

$$m \equiv 1 \mod 13 \text{ and } m \text{ divides } 25.$$

This implies $m = 1$. So there is one subgroup K of order 13 in G.

Since the orders of H and K are relatively prime, we have that $H \cap K = \{e\}$. They are both Normal subgroups of G. As order of G is 25×13, $G = HK$. Hence G is a semidirect product of H and K.

Now as order of K is 13, K is cyclic and isomorphic to $\mathbb{Z}_{13}$. Similarly H can be isomorphic to either $\mathbb{Z}_5 \times \mathbb{Z}_5$ or $\mathbb{Z}_{25}$. Hence upto isomorphism the groups of order 325 are

$$\mathbb{Z}_{13} \times \mathbb{Z}_5 \times \mathbb{Z}_5 \quad \text{or} \quad \mathbb{Z}_{13} \times \mathbb{Z}_{25}.$$

Case 2: $26 = 2 \times 13$.

If G is abelian then the only group upto isomorphism is C_{26}. Suppose G is not abelian. Let H be a 13-Sylow subgroup. As order of H is 13 a prime, it is cyclic with generator a (say). H is normal in G (why ?).
Let $K = \{1, b\}$ be a 2-Sylow subgroup. Note that $ba \in G$, is not an element of H since $b \notin H$ and in particular $ba \neq e$, where e is the identity. Further, ba cannot have order 13 since H is the only subgroup of order 13. It also cannot have order 26 since G is not abelian (in particular not cylic as well). Thus $(ba)^2 = 1$ (why ?).
As

$$a^{13} = b^2 = (ab)^2 = e,$$

we can conlude that G is isomorphic to the dihedral group with presentation

$$\langle r, s | r^{13} = s^2 = (sr)^2 = e \rangle$$

by allowing the isomorphism ϕ to be defined from

$$\phi(a) = r, \phi(b) = s$$

It also easy to show that G is a semidirect product of $\mathbb{Z}_2$ and $\mathbb{Z}_{13}$ directly (as we have shown above) or from the fact that dihedral group is isomorphic to it.