Writing of Mathematics **Quiz 7** **Semester I 2018/19**

Your name: **Solution** October 29th, 2018

1.Supppse $\{x_n\}_{n=1}^{\infty}$and $\{y_n\}_{n=1}^{\infty}$are given by

$$x_n = \left(1+\frac{1}{n}\right)^n \quad \text{and} \quad y_n = \sum_{k=1}^{n} \frac{1}{k!}$$

for all $n \in \mathbb{N}$.

Assume that:

(i) $\{y_n\}_{n=1}^{\infty}$converges to a number (say) e

(ii) $x_n \leq y_n$ for all $n \geq 1$.

For all $m \geq 1$, show that

$$y_m \leq \liminf_{n\to\infty} x_n,$$

and conclude that $\{x_n\}_{n=1}^{\infty}$also converges to e.

Solution: Let $m \geq 1$ be fixed. Observe that, using the Binomial Expansion,

$$x_n = \sum_{k=0}^{n} \binom{n}{k} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{\prod_{i=0}^{k-1}(n-i)}{k!} \frac{1}{n^k} = \sum_{k=0}^{n} \frac{1}{k!} \prod_{i=0}^{k-1}(1-\frac{i}{n}).$$

Note that all terms inside the above summand are non-negative. Therefore for $n \geq m$,

$$\begin{aligned} x_n &= \sum_{k=0}^{m} \frac{1}{k!} \prod_{i=0}^{k-1}(1-\frac{i}{n}) + \sum_{k=m+1}^{n} \frac{1}{k!} \prod_{i=0}^{k-1}(1-\frac{i}{n}). \\ &\geq \sum_{k=0}^{m} \frac{1}{k!} \prod_{i=0}^{k-1}(1-\frac{i}{n}). \end{aligned}$$

For $\frac{1}{n} \to 0$ as $n \to \infty$, this implies that for any $1 \leq i \leq m$

$$\frac{i}{n} \to 0 \text{ as } n \to \infty.$$

So, by algebra of limits,

$$\alpha_n := \sum_{k=0}^{m} \frac{1}{k!} \prod_{i=0}^{k-1}(1-\frac{i}{n}) \to \sum_{k=0}^{m} \frac{1}{k!} \text{ as } n \to \infty.$$

Now $x_n \geq \alpha_n$ for all $n \geq m$, therefore

$$\liminf_{n\to\infty} x_n \geq \liminf_{n\to\infty} \alpha_n = \sum_{k=0}^{m} \frac{1}{k!} = y_m. \tag{1}$$

From Assumption (ii) we have that for all $n \geq 1$,

$$x_n \leq y_n$$

which, along with Assumption (i), implies that

$$\limsup_{n\to\infty} x_n \leq \limsup_{n\to\infty} y_n = e. \tag{2}$$

In(1), letting $m \to \infty$ on both sides we have

$$\liminf_{n\to\infty} x_n \geq \liminf_{n\to\infty} \alpha_n = \liminf_{m\to\infty} y_m = e. \tag{3}$$

As

$$\liminf_{n\to\infty} x_n \leq \limsup_{n\to\infty} x_n$$

from (2) and (3) we have that

$$\lim_{n\to\infty} x_n = e.$$

$\square$

2. Suppose two teams play a series of games, each producing a winner and a loser, until one team has won two more games than the other. Let G be the total number of games played. Assuming each team has chance 0.5 to win each game, independent of results of the previous games. Find the expected value of G.