Your name: **Solution:** October 25th, 2018

Proof of Assumptions:

(a) Show that $1 - a \leq e^{-a}$ for any $a \in [0, 1)$.
Solution: For $x \in \mathbb{R}$, we know

$$e^x = E(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

is well defined and

$$E(x + y) = E(x)E(y).$$

Consequently, for $h \neq 0$ and $x \in \mathbb{R}$

$$\frac{E(x+h) - E(x)}{h} - E(x) = E(x)\left(\frac{E(h) - 1 - h}{h}\right)$$

For $0 <| h |< 1$,

$$|\sum_{n=2}^{\infty} \frac{h^n}{n!}| \leq \sum_{n=2}^{\infty} \frac{| h |^n}{n!} = E(| h |) < \infty \qquad \text{(why ?).}$$

So we can rewrite

$$\left|\frac{E(h) - 1 - h}{h}\right| = \left|\frac{1}{h}\sum_{n=2}^{\infty} \frac{h^n}{n!}\right| \leq \sum_{n=2}^{\infty} \frac{| h |^{n-1}}{n!} \leq | h | \sum_{n=2}^{\infty} \frac{1}{n!} < | h | e,$$

for $0 <| h |< 1$. Therefore

$$\lim_{h \to 0} \frac{E(x+h) - E(x)}{h} - E(x) = E(x).$$

Hence $E'(x) = E(x)$ for all $x \in \mathbb{R}$. If $g : (0, 1) \to \mathbb{R}$ is given by

$$g(a) = 1 - a - E(-a)$$

then g is differentiable and by chain rule

$$g'(a) = -1 + E(-a) \quad \leq 0, \quad \forall a \in (0, 1). \qquad \text{(why ?)}$$

Therefore for all $0 \leq a < 1$, $g(a) \leq g(0)$ which implies

$$1 - a \leq e^{-a}.$$

□

(b) Let $\{a_n\}_{n \geq 1}$ be a sequence of numbers such $0 \leq a_n < 1$. Using induction, show that

$$\prod_{k=1}^{n} (1 - a_k) \geq 1 - \sum_{k=1}^{n} a_k.$$

Solution: For $n \geq 1$, let

$$P(n) \equiv \prod_{k=1}^{n} (1 - a_k) \geq 1 - \sum_{k=1}^{n} a_k.$$

Step 1: As $1 - a_1 = 1 - a_1$ so $P(1)$ is true.
Step 2: Assume for $P(m)$ is true. So,

$$\prod_{k=1}^{m} (1 - a_k) \geq 1 - \sum_{k=1}^{m} a_k.$$

Step 3: Let $n = m + 1$, we have

$$\prod_{k=1}^{m+1}(1-a_k) = \left(\prod_{k=1}^{m}(1-a_k)\right)(1-a_{m+1}) \geq (1-\sum_{k=1}^{m} a_k)(1-a_{m+1})$$
$$= 1 - \sum_{k=1}^{m} a_k - a_{m+1} + a_{m+1}\sum_{k=1}^{m} a_k \geq 1 - \sum_{k=1}^{m+1} a_k.$$

So $P(m+1)$ is true. Hence by the principle of induction, $P(n)$ is true for all $n \geq 1$. □

1. Let $\{a_n\}_{n\geq 1}$ be a sequence of numbers such $0 < a_n < 1$. For any $n \geq 1$, let $b_n = \prod_{i=1}^{n}(1-a_i)$. Show that

(i) b_n converges to $b \in (0,1)$ if $\sum_{k=1}^{\infty} a_k < \infty$.

Solution: Note that $0 < b_n < 1$. Observe by the same proof as part (b) using induction we have that for all $m \geq 1$ and , $n > m$,

$$\prod_{k=m}^{n}(1-a_k) \geq 1 - \sum_{k=m}^{n} a_k.$$

As $\sum_{k=1}^{\infty} a_k < \infty$, there is a $N > 0$ such that

$$\sum_{k=N}^{\infty} a_k < \frac{1}{2}.$$

As $0 \leq a_n$, for $n \geq N$, we have

$$\sum_{k=N}^{n} a_k < \frac{1}{2}$$

and

$$\frac{b_n}{b_{N-1}} = \prod_{k=N}^{n}(1-a_k) \geq 1 - \sum_{k=N}^{n} a_k \geq 1 - \frac{1}{2} = \frac{1}{2}.$$

So if $B = \min\{b_i : 1 \leq i \leq N-2\}$ then

$$b_n \geq \min\{B, \frac{b_{N-1}}{2}\} > 0$$

for all $n \geq 1$. Further, $b_n \geq b_{n+1}$ for all $n \geq 1$, i.e. it is monotonically decreasing. Therefore b_n converges to a positive number $b > \geq \min\{B, \frac{b_{N-1}}{2}\}$ which implies $b\ in(0,1)$ □

(ii) b_n converges to 0 if $\sum_{k=1}^{\infty} a_k = \infty$.

Solution: Note that $0 \leq b_n \leq 1$. Observe by part(a), $1 - a_i \leq e^{-a_i}$ for all $1 \leq i$. Hence by induction we have for all $k \geq 1$,

$$0 \leq b_k = \prod_{i=1}^{k}(1-a_i) \leq \prod_{i=1}^{k} e^{-a_i} = e^{-\sum_{i=1}^{k} a_i} = e^{-x_k},$$

where $x_k = \sum_{i=1}^{k} a_i$.

As $x_k \to \infty$ as $k \to \infty$, by Problem 1 (d) in Hw 9, we know that $e^{-x_k} \to 0$ as $k \to \infty$.

Therefore by the Squeeze theorem we have

$$\lim_{k\to\infty} b_k = 0.$$

□