Writing of Mathematics **Quiz 3** **Semester I 2018/19**

Your name: **Solution** October 11th, 2018

1. Let G be a set endowed with an operation $\cdot$. It is given that there is an $e \in G$ such that $g \cdot e = g$ for all $g \in G$ and for all $g \in G$ there is a $h \in G$ such that $h \cdot g = e$. Is $(G, \cdot)$ necessarily a group ?

Solution: Consider

$$G = \{\begin{pmatrix} a & a \\ b & b \end{pmatrix} : a > 0, b \geq 0, a, b \in \mathbb{Q}\},$$

with usual matrix multiplication. That is

$$\begin{pmatrix} a & a \\ b & b \end{pmatrix} \cdot \begin{pmatrix} c & c \\ d & d \end{pmatrix} = \begin{pmatrix} ac + ad & ac + ad \\ bc + bd & bc + bd \end{pmatrix}$$

We define $e = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$. Clearly $g \cdot e = g$ for all $g \in G$.

Also for $g \in G$ with $g = \begin{pmatrix} a & a \\ b & b \end{pmatrix}$, let $h = \begin{pmatrix} \frac{1}{a+b} & \frac{1}{a+b} \\ 0 & 0 \end{pmatrix}$.

We note that $h \cdot g = e$. As

$$g \cdot h = \begin{pmatrix} \frac{a}{a+b} & \frac{a}{a+b} \\ \frac{b}{a+b} & \frac{b}{a+b} \end{pmatrix},$$

it does not have an inverse.

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Your name: **Solution** October 11th, 2018

1. Let G be a set endowed with an associative $\cdot$ operation. It is given that there is an $e \in G$ such that $g \cdot e = g$ for all $g \in G$ and for all $g \in G$ there is a $h \in G$ such that $g \cdot h = e$. Is $(G, \cdot)$ necessarily a group ?

Solution: It is given that the operation $\cdot$ is associative. Fix $g \in G$. Let $h \in G$ be such that $g \cdot h = e$. For this $h \in G$ there is a $k \in G$ such that $h \cdot k = e$. Hence

$$h \cdot g = (h \cdot g) \cdot e = (h \cdot g) \cdot (h \cdot k) = h \cdot (g \cdot h) \cdot k = h \cdot e \cdot k = h \cdot k = e$$

Using the above and $g \cdot e = g$, we have

$$e = g \cdot h \cdot g = g \cdot e = g.$$

Since $g \in G$ was arbitrary we have shown that there is an $e \in G$ such $g \cdot e = g = e \cdot g, \forall g \in G$, and for all $g \in G$ there is a $h \in G$ such that $g \cdot h = e = h \cdot g$.

So G is a group.