1. (*Exponential function : e^x*) Consider the function $E : \mathbb{R} \to \mathbb{R}$ given by $E(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}$.
   (a) Show that E is well-defined and $E(x+y) = E(x)E(y)$, for all $x, y \in \mathbb{R}$.
   (b) Show that E is a continuous and monotonically increasing (strictly) function on $\mathbb{R}$.
   (c) Let $e = E(1)$. Show that $E(x) = e^x$ for all $x \in \mathbb{R}$.
   (d) Show that $\lim_{x\to\infty} x^n e^{-x} = 0$ for all $n \in \mathbb{N}$.

Solution:(a)
Proof of Well-definedness: Let $x \in \mathbb{R}$. Applying the ratio test to the given series, with $a_n = \frac{x^n}{n!}$ we have

$$\limsup_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+1)!}}{\frac{x^n}{n!}} \right| = \limsup_{n\to\infty} \left| \frac{x}{n} \right| = |x| \lim_{n\to\infty} \frac{1}{n} = 0$$

Therefore the series $\sum_{n=1}^{\infty} \frac{x^n}{n!}$ converges absolutely for all $x \in \mathbb{R}$. The function E is well defined.
Proof of Product-sum Formula: For any given $x, y \in \mathbb{R}$, define, for all $n \in \mathbb{N} \cup \{0\}$,

$$a_n = \frac{x^n}{n!}, \; b_n = \frac{y^n}{n!}, \; c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

Observe that

$$c_n = \sum_{k=0}^{n} a_k b_{n-k} = \sum_{k=0}^{n} \frac{x^k y^{n-k}}{k!(n-k)!} = \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = \frac{(x+y)^n}{n!}$$

We know that

$$\sum_{n=0}^{\infty} a_n, \; \sum_{n=0}^{\infty} b_n \text{ and } \sum_{n=0}^{\infty} c_n \text{ converge absolutely to } E(x), E(y), \text{ and } E(x+y) \text{ respectively.}$$

However, by the Cauchy product theorem,

$$\sum_{n=0}^{\infty} c_n \text{ converges to } E(x)E(y).$$

Therefore for $x, y \in \mathbb{R}$ we have

$$E(x)E(y) = E(x+y) \tag{1}$$

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Solution: (b)
Proof of Strictly increasing: $E(0) = 1$. For $x > 0$, as

$$\sum_{k=0}^{n} \frac{x^k}{k!} \geq 1 + x, \; \forall n \geq 1 \qquad \text{this implies} \qquad E(x) \geq 1 + x > 1.$$

If $x < 0$, then $-x > 0$ and by above $E(-x) > 1$. Further, $1 = E(0) = E(x + (-x)) = E(x)E(-x)$. So for $x < 0$,

$$E(x) = \frac{1}{E(-x)} \in (0, 1)$$

. If $a > b \in \mathbb{R}$, then $a = b + a - b$ with $a - b > 0$.

$$E(a) = E(b + (a-b)) = E(b)E(a-b) > E(b) \cdot 1.$$

So $a > b \implies E(a) > E(b)$, and E is a strictly increasing function on $\mathbb{R}$.

Proof of Continuity: Let $a \in \mathbb{R}$ be given. Let $M > 0$ such that $a \in (-M, M)$. For any $b \in (-M, M)$ and $n \in \mathbb{N}$, we have

$$|a^n - b^n| = \left|(a-b)\sum_{k=0}^{n-1} a^k b^{n-k-1}\right| \leq |a-b|\sum_{k=0}^{n-1}|a|^k|b|^{n-k-1} < |a-b|\big(nM^{n-1}\big)$$

So, for $k \geq 1$, using the above, we have

$$\left|\sum_{n=1}^{k}\frac{a^n - b^n}{n!}\right| \leq \sum_{n=1}^{k}\frac{|a^n - b^n|}{n!} \leq |a-b|\sum_{n=1}^{k}\frac{nM^{n-1}}{n!} \leq |a-b|\sum_{n=0}^{k}\frac{M^n}{n!} \tag{2}$$

Note that

$$|E(a) - E(b)| = \left|\sum_{n=1}^{\infty}\frac{a^n - b^n}{n!}\right| \text{ and } E(M) = \sum_{n=0}^{\infty}\frac{M^n}{n!}$$

The first equality uses the fact that the sum (or difference) of two absolutely convergent series is absolutely convergent and converges to the sum (or difference) of the limits. The second equality follows from definition. Using the above with (2) we have for $a, b \in (-M, M)$

$$|E(a) - E(b)| \;\; \leq \;\; |\; a - b \;|\; E(M) \tag{3}$$

Let $\epsilon > 0$ be given. Choose

$$\delta = \frac{1}{2}\min\left\{|\; a - M \;|, |\; a + M \;|, \frac{\epsilon}{E(M) + 1)}\right\}.$$

Let $c \in \mathbb{R}$ such that $|c - a| < \delta$. By choice of $\delta > 0$ $c \in (-M, M)$. From (3) we have

$$|E(a) - E(c)| \leq |a - c|E\big(M\big) < \delta E\big(M\big) < \epsilon$$

As $\epsilon > 0$ was arbitrary. E is continuous at a. As $a \in \mathbb{R}$ was arbitrary, E is continuous.

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Solution:(c)
$E(0) = 1 = e^0$. If $x \in \mathbb{N}$,

$$E(x) = E\left(\sum_{k=1}^{x} 1\right) = \prod_{k=1}^{x} E(1) = e^x$$

If $x = 1/n$, for $n \in \mathbb{N}$, then

$$e = E(1) = E\left(\sum_{k=1}^{n} x\right) = \prod_{k=1}^{n} E(x) = \big(E(x)\big)^n$$

$$\text{As } E(x) > 0,\;\; E(x) = e^{\frac{1}{n}} = e^x$$

If $x \in \mathbb{Q}$, let $x = p/q$, where $q \in \mathbb{N}$ and $p \in \mathbb{Z}$. If $p = 0$, then $x = 0$ and $E(x) = e^x$. If $p > 0$,

$$E(x) = E\left(\frac{p}{q}\right) = E\left(\sum_{k=1}^{p}\frac{1}{q}\right) = \prod_{k=1}^{p} E(1/q) = \big(e^{1/q}\big)^p = e^x$$

If $p < 0$, then $-x = \frac{-p}{q}$ and

$$E(x) = \frac{1}{E(-x)} = \frac{1}{e^{-x}} = e^x$$

Thus,

$$\text{for all } x \in \mathbb{Q},\; E(x) = e^x.$$

Proof of formula for $x \in \mathbb{R}$: We will now show the equality for all $x \in \mathbb{R}$. Since $e > 1$, for[1] $x \in \mathbb{R}$,

$$e^x := \sup\{e^r : r \in \mathbb{Q}, r < x\} = \sup\{E(r) : r \in \mathbb{Q}, r < x\}.$$

Let $S = \{E(r) : r \in \mathbb{Q}, r < x\}$.
As E is monotonically increasing,

$$E(x) \text{ is an upper bound of } S. \tag{4}$$

Let $\epsilon > 0$ be given. By continuity of E,

$$\text{there exists } \delta > 0 \text{ such that } |y - x| < \delta \implies |E(x) - E(y)| < \epsilon.$$

This implies

$$\text{if a rational } r \in (x - \delta, x) \text{ then } E(r) > E(x) - \epsilon.$$

But $E(r) \in S$. Therefore,

$$E(x) - \epsilon \text{ is not an upper bound of } S. \tag{5}$$

From (4) and (5) we have that $E(x) = \sup S = e^x$ for all $x \in \mathbb{R}$.

Solution:(d)
Proof of decay rate: For $x > 0$ and $n \in \mathbb{N}$,

$$\sum_{k=0}^{M} \frac{x^k}{k!} \geq \frac{x^{n+1}}{(n+1)!}$$

whenever $M > n$. Taking limits as $M \to \infty$ we have that for all $x > 0, n \geq 1$,

$$\frac{e^x}{x^n} \geq \frac{x}{(n+1)!}$$

This implies that, for $x > 0$ and $n \in \mathbb{N}$

$$0 < \frac{x^n}{e^x} \leq \frac{(n+1)!}{x} \tag{6}$$

Fix $n \geq 1$. Let $\epsilon > 0$ be given. Then $\exists M > 0$ such that

$$\frac{(n+1)!}{x} < \epsilon \text{ whenever } x > M.$$

Hence, using (6) and above we have for $x > M$

$$0 < \frac{x^n}{e^x} < \epsilon.$$

As $\epsilon > 0$ was arbitrary, we have shown that

$$\lim_{x \to \infty} x^n e^{-x} = 0.$$

[1] See chapter 1 exercise 6 of Rudin, Principle of Mathematical Analysis

2 (*Logarithm function:- $L(x)$*) Let E be the function as defined in the previous question. Let $L : (0,\infty) \to \mathbb{R}$ such that

$$L(E(y)) = y, \forall y \in \mathbb{R}$$

(a) Show that L is well-defined and $L(uv) = L(u) + L(v)$, for all $u, v \in (0,\infty)$. ($L(x)$ is denoted by $\ln(x)$ for all $x > 0$)

(b) Show that L is a continuous monotonically increasing (strictly) function.

(c) Show that for any $\alpha \in \mathbb{R}$, $x \in [0,\infty)$, $x^\alpha = E(\alpha(L(x))) = e^{\alpha L(x)}$.

Solution: (a)
We need to show that E is a bijection from $\mathbb{R}$ onto $(0,\infty)$. This will imply that L, given by $L(E(y)) = y$, is well-defined.

Proof of E is One to one: From Problem 1(b), we know that E is strictly increasing. So

$$\text{if} \qquad y_1 < y_2 \qquad \text{then} \qquad E(y_1) < E(y_2). \tag{7}$$

Hence E is a $1-1$ function. From definition, we have

$$E(x) > 0 \text{ for all } x > 0. \tag{8}$$

Secondly, from Problem 1(d) we can conclude that

$$\lim_{x\to-\infty} E(x) = 0 \qquad \text{(why ?)}. \tag{9}$$

From (8), (9), and (7) we have that $\inf_{x\in\mathbb{R}} E(x) \geq 0$ which implies Range$(E) \subseteq (0,\infty)$.

Proof of Range $(E) = (0,\infty)$: Let $a \in (1,\infty)$. Note that $E(0) = 1$ and $E(a) = \sum_{k=0}^{\infty} \frac{a^k}{k!} > 1 + a$. From, Problem 1(b), we know that E is continuous. So, by the Intermediate value theorem,

$$\text{if } a > 1, \text{ there exists } b \in (0,a) \text{ such that } E(b) = a. \tag{10}$$

Let $0 < a < 1$, then $\frac{1}{a} > 1$ by (10) there exists $c \in (0, \frac{1}{a})$ such that $E(c) = \frac{1}{a}$. Using Problem 1(c), if $d = -c$ then

$$E(d) = a. \tag{11}$$

From (10) and (11) we have, $(0,\infty) \subseteq$ Range(E). So Range$(E) = (0,\infty)$.

Proof of property of L : Let $u, v \in (0,\infty)$ and let $s = L(u)$ and $t = L(v)$. So by definition, $u = E(s)$ and $v = E(t)$. Then, by Problem 1(a),

$$uv = E(s)E(t) = E(s+t).$$

Hence

$$L(uv) = L(E(s+t)) = s + t = L(u) + L(v).$$

Solution: (b)
Proof of Strictly increasing: Let $x, y \in (0, \infty)$ such that $x < y$. Let $u = L(x)$ and $v = L(y)$. Then, $x = E(u)$ and $y = E(v)$. As E is strictly increasing, if $u \geq v$ then $x = E(u) \geq E(y) = y$, which is not true as $x < y$. Therefore $u < v$, and hence L is strictly increasing.

Proof of Continuity: We will now show that L is continuous. Let $a \in (0, \infty)$ and $b = L(a)$. Let $\epsilon > 0$ be given. Define

$$\delta := \min\big\{E(b+\epsilon) - E(b), E(b) - E(b-\epsilon)\big\}.$$

As E is strictly increasing, $\delta > 0$. Let $c \in (0, \infty)$ such that $|c - a| < \delta$. Then

$$\begin{aligned} & c \in (a-\delta, a+\delta), \\ \text{as } E(b) = a, \Longrightarrow \quad & c \in (E(b)-\delta, E(b)+\delta) \\ \text{by definition of } \delta, \Longrightarrow \quad & c \in (E(b-\epsilon), E(b+\epsilon)). \end{aligned}$$

As L is strictly increasing, the above implies that

$$L(c) \in (b-\epsilon, b+\epsilon).$$

So we have shown that

$$\text{if } \mid c - a \mid < \delta \text{ then } |L(c) - L(a)| < \epsilon.$$

As $\epsilon > 0$ was arbitrary. L is continuous at a. As $a \in (0, \infty)$ was arbitrary, L is continuous[2].

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Solution:(c)
Proof of formula: Let $x \in (0, \infty)$. Note that

$$L(E(x)) = x = E(L(x)).$$

Then

$$\begin{aligned} E(\alpha L(x)) &= e^{\alpha L(x)} && \text{(By Problem 1(c))} \\ &= \left(e^{L(x)}\right)^{\alpha} && \text{(why ?)} \\ &= \big(E(L(x))\big)^{\alpha} && \text{(By Definition)} \\ &= x^{\alpha} && \text{(By above).} \end{aligned}$$

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[2] Have we used anywhere that E is continuous in this proof ?.

2. (*Exercise towards Verifying Convergence in Gradient Descent*) Let $\{a_n\}_{n\geq 1}$ be a sequence of numbers such $0 \leq a_n < 1$.

   (a) Using induction, show that $\prod_{i=1}^{n}(1-a_i) \geq 1 - \sum_{k=1}^{n} a_k$.

   (b) Show that $1 - a \leq e^{-a}$ for any $a \in [0,1)$.

   (c) For any $n \geq 1$, let $b_n = \prod_{i=1}^{n}(1-a_i)$.

      i. Show that b_n converges to 0 if $\sum_{k=1}^{\infty} a_k = \infty$.

      ii. Show that b_n converges to $b \in (0,1)$ if $\sum_{k=1}^{\infty} a_k < \infty$.

   Solution: See Quiz 7 Solution.

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3. Let $\{a_n\}_{n\geq 1}$ be a bounded sequence. Then show that it has a subsequence convergent in $\mathbb{R}$.
   Solution: See Theorem 3.4.8 *Bartle and Sherbert, Introduction to Real Analysis.*

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4. (*Finding Roots of a number*) Let $a > 0$ and choose $s_1 > \sqrt{a}$. Define

$$s_{n+1} := \frac{1}{2}(s_n + \frac{a}{s_n})$$

   for $n \in \mathbb{N}$.

   (a) Show that s_n is monotonically decreasing and $\lim_{n\to} s_n = \sqrt{a}$.

   (b) If $z_n = s_n - \sqrt{a}$ then show that $z_{n+1} < \frac{z_n^2}{2\sqrt{a}}$.

   (c) Let $f(x) = x^2 - a$. Show that $s_n = s_{n-1} - \frac{f(s_{n-1})}{f'(s_{n-1})}$.

   (d) Draw graph of f with $a = 4$ and plot the sequence s_n for a few steps when $s_0 = 5$.

5. Prove that if G is an *abelian* group of order pq, where p and q are distinct primes, then G is cyclic.
   Solution: See Theorem 15.10 in *Abstract Algebra Theory and Applications Thomas W. Judson and Robert A. Beezer*

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6. Classify all groups of order 325 and 26.
   Solution: See Quiz 8 Solution.

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