Homework 9
Problem Due: 2,
Due Date: 25th, October 2018.

1. (**2-dimension Linear Least Squares**)Suppose we believe that a variable z is dependent on two variables x, y via a linear relationship $z = ax + by + c$, and we are given n data points : $\left\{ \left(\begin{bmatrix} x_i \\ y_i \end{bmatrix}, z_i \right) : 1 \leq i \leq n \right\}$. How would you proceed to find a, b, c so as to minimize:
$$\sum_{i=1}^{n} (z_i - ax_i - by_i - c)^2?$$

2. Let $\mathbb{X} = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ equipped with Binary addition structure. Consider the XOR (exclusive OR function) on $\mathbb{X}$, i.e
$$\text{XOR}\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) = 0, \quad \text{XOR}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = 1, \quad \text{XOR}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = 1, \quad \text{XOR} = \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = 0.$$
The above is the true relationship but you are not told that. You are given the following data set of $\left(\begin{bmatrix} x \\ y \end{bmatrix}, z \right)$,
$$\left\{ \left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, 0 \right), \quad \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}, 1 \right), \quad \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, 1 \right), \quad \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, 0 \right) \right\}$$
   (a) Assume z is a linear function of elements in $\mathbb{X}$. Find best least square linear function.
   (b) Let
$$W = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, w = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, c = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, b = 0$$
   and
$$h\left(\begin{bmatrix} x \\ y \end{bmatrix} \right) = w^T \left(\max\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, W^T \begin{bmatrix} x \\ y \end{bmatrix} + c \right\} \right) + b$$
      i. Evaluate $h\left(\begin{bmatrix} x \\ y \end{bmatrix} \right)$ for $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{X}$
      ii. Evaluate $\sum_{i=1}^{4} \left(z_i - h\left(\begin{bmatrix} x_i \\ y_i \end{bmatrix} \right) \right)^2$
   (c) In the previous question : can you device a procedure by which you can find W, w, c, b ?

3. Let $f(x, y) = \frac{x^2}{2} + 2y^2$
   (a) Can you guess a minima for f ?
   (b) Draw the level curves of f at levels $1, 10, 100$.
   (c) Let $z^{(0)} = (4, 1)$. Find a suitable t_k for $k \geq 1$. Calculate $z^{(k)}$ using Gradient Descent algorithm.
   (d) Does $z^{(k)}$ converge and if so where ?

[1]**Office hours:** I will be in my office from 8:15-9am on Tue,Wed, Thu and from 2-3pm on Wed to answer any questions that you may have. Please feel free to drop by during these times to clarify any doubts that you may have.

Random Connections

1. *(Exponential function: e^x)* Consider the function $E : \mathbb{R} \to \mathbb{R}$ given by $E(x) = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}$.
   (a) Show that E is well-defined and $E(x + y) = E(x)E(y)$, for all $x, y \in \mathbb{R}$.
   (b) Show that E is a continuous and monotonically increasing (strictly) function on $\mathbb{R}$.
   (c) Let $e = E(1)$. Show that $E(x) = e^x$ for all $x \in \mathbb{R}$.
   (d) Show that $\lim_{x\to\infty} x^n e^{-x} = 0$ for all $n \in \mathbb{N}$.
2. (*Logarithm function:-* $\ln(x)$) Let E be the function as defined in the previous question. Let $L : (0, \infty) \to \mathbb{R}$ such that
   $$L(E(y)) = y, \forall y \in \mathbb{R}$$
   (a) Show that L is well-defined and $L(uv) = L(u) + L(v)$, for all $u, v \in (0, \infty)$. ($L(x)$ is denoted by $\ln(x)$ for all $x > 0$)
   (b) Show that L is a continuous monotonically increasing (strictly) function.
   (c) Show that for any $\alpha \in \mathbb{R}$, $x \in [0, \infty)$, $x^\alpha = E(\alpha(\ln(x))) = e^{\alpha \ln(x)}$.
3. (*Exercise towards Verifying Convergence in Gradient Descent*) Let $\{a_n\}_{n\geq 1}$ be a sequence of numbers such $0 \leq a_n < 1$.
   (a) Using induction, show that $\prod_{i=1}^{n}(1 - a_i) \geq 1 - \sum_{k=1}^{n} a_k$.
   (b) Show that $1 - a \leq e^{-a}$ for any $a \in [0, 1)$.
   (c) For any $n \geq 1$, let $b_n = \prod_{i=1}^{n}(1 - a_i)$.
      i. Show that b_n converges to 0 if $\sum_{k=1}^{\infty} a_k = \infty$.
      ii. Show that b_n converges to $b \in (0, 1)$ if $\sum_{k=1}^{\infty} a_k < \infty$.
4. Let $\{a_n\}_{n\geq 1}$ be a bounded sequence. Then show that it has a subsequence convergent in $\mathbb{R}$.
5. (*Finding Roots of a number*) Let $a > 0$ and choose $s_1 > \sqrt{a}$. Define
   $$s_{n+1} := \frac{1}{2}(s_n + \frac{a}{s_n})$$
   for $n \in \mathbb{N}$.
   (a) Show that s_n is monotonically decreasing and $\lim_{n\to} s_n = \sqrt{a}$.
   (b) If $z_n = s_n - \sqrt{a}$ then show that $z_{n+1} < \frac{z_n^2}{2\sqrt{a}}$.
   (c) Let $f(x) = x^2 - a$. Show that $s_n = s_{n-1} - \frac{f(s_{n-1})}{f'(s_{n-1})}$.
   (d) Draw graph of f with $a = 4$ and plot the sequence s_n for a few steps when $s_0 = 5$.
6. Prove that if G is an *abelian* group of order pq, where p and q are distinct primes, then G is cyclic.
7. Classify all groups of order 325 and 26.

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