Homework 6
Due Date: October 4th, 2018
Problems Due: 2 (c), 4

1. Let A, B, C be non-empty sets. Let $f : A \to B$, $g : B \to C$ and $h : A \to C$ given by $h = g \circ f$. Decide which of the following statements are true or false.
   (a) If f and g are injective, then h is injective.
   (b) If f and g are surjective, then h is surjective.
   (c) If h is injective, then f is injective.
   (d) If h is injective, then g is injective.
   (e) If h is surjective, then f is surjective.
   (f) If h is surjective, then g is surjective.
2. Decide if Card $(A) <$ Card (B) , Card $(A) >$ Card (B), or Card $(A) =$ Card (B) when
   (a) $A = (a, b)$ and $B = [a, b]$ for some $a, b \in \mathbb{R}$.
   (b) $A = \mathbb{N}$ and $\mathbb{Z}$.
   (c) $A = \mathbb{N}$ and $\mathbb{Z} \times \mathbb{Z}$
   (d) $A = \mathbb{Z} \times \mathbb{Z}$ and $B = \mathbb{Q}$.
3. Let $n \in \mathbb{N}$ and $k < n$. Show that
   (a) $\sum_{i=0}^{n} \binom{i}{k} = \binom{n+1}{k+1}$
   (b) Use the above to derive formulae for
   $$\sum_{i=0}^{n} i \text{ and } \sum_{i=0}^{n} i^2$$
4. Find the number of transpositions needed to convert a permtuation of $\{1, 2, 3, \ldots, n\}$ with k cycles to the identity.

Extra Credit Puzzles:

1. The class teacher of B.Math (hons.) has to decide on Siva's request of holding a class at 3am every other Saturday on non-leap years. He delegates it to a student committee of 23, with a designated chairperson. Then Siva comes to class of 40 students and ask the following question. There are two ways to do this:
   (a) Select 23 people from the class and then choose a chairperson from the selected people OR
   (b) Select a chairperson from the class and then fill out the rest of the committee.

   Find the number of ways you can do (a) and (b). Can you use the problem to obtain a general combinatorial identity with n students and k person committee with one designated chair person ?
2. How many regions are created by n lines in the plane such that no two lines are parallel and no three lines intersect at the same point?

[2]**Office hours:** I will be in my office from 8:15-9am on Tue,Wed, Thu and from 2-3pm on Wed to answer any questions that you may have. Please feel free to drop by during these times to clarify any doubts that you may have.

Serie(ou)s Entanglements and Lawless Large numbers

1. Suppose $\{z_n\}_{n\geq 1}$ is a sequence of real numbers. Decide whether the series $\sum_{n=1}^{\infty} z_n$ converges in each of the following cases:

$$\begin{aligned} &\text{(i)} \quad z_n = \frac{\sqrt{n}}{2n^3-1} \\ &\text{(ii)} \quad z_n = \left(\frac{n}{2n+1}\right)^n \\ &\text{(iii)} \quad z_n = \frac{n^2-n+1}{n^3+1} \end{aligned}$$

2. Let $x \in [0,1]$. Show that there is a sequence $\{b_n : n \in \mathbb{N}\}$, such that $b_n \in \{0,1\}$ and

$$\sum_{k=1}^{n} \frac{b_k}{2^k} \leq x \leq \sum_{k=1}^{n} \frac{b_k}{2^k} + \frac{1}{2^n}, \tag{1}$$

for all $n \in \mathbb{N}$. Conversely given a sequence $\{b_n : n \in \mathbb{N}\}$ such that $b_n \in \{0,1\}$ there is a $x \in [0,1]$ satisfying (1) for all $n \in \mathbb{N}$.

3. Suppose $\{z_n\}_{n\geq 1}$ is a sequence of real numbers given by

$$z_n = \begin{cases} 0 & \text{if } n = 1,2 \\ \frac{(-1)^{n+1}}{n} & \text{if } n \geq 3 \end{cases}$$

   (a) Decide whether the series $\sum_{n=1}^{\infty} z_n$ converges absolutely.

   (b) Decide whether the series $\sum_{n=1}^{\infty} z_n$ converges.

   (c) Show that there is an $f : \mathbb{N} \to \mathbb{N}$, an injective function such that if $T_n = \sum_{k=1}^{n} a_{f(k)}$ then

$$\liminf_{n\to\infty} T_n = -\frac{1}{2} \text{ and } \limsup_{n\to\infty} T_n = \frac{1}{2}$$

4. Suppose $\{z_n\}_{n\geq 1}$ be a sequence of real numbers such that $z_n \in \{0,1\}$.

   (a) Let $0 < \alpha < \beta < 1$. Show that there is a sequence $z : \mathbb{N} \to \{0,1\}$, such that if

$$T_n = \frac{1}{n}\sum_{k=1}^{n} z_k$$

   then

$$\liminf_{n\to\infty} T_n = \alpha \text{ and } \limsup_{n\to\infty} T_n = \beta$$

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