Homework 5
Due Date: September 27th, 2018
Problems Due: 1(a) and 2(a).

1. Using the method of induction prove that
   (a) $\sum_{i=0}^{n} i!i = (n+1)! - 1$
   (b) For all $x > 0$, $(1+x)^n \geq 1 + nx$
   (c) Suppose for $1 \leq i \leq n$, a_i are real numbers. Show that
   $$\mid \sum_{i=1}^{n} a_i \mid \leq \sum_{i=1}^{n} \mid a_i \mid$$
2. Using the method of strong induction prove that
   (a) Let $\{x_n\}_{n=1}^{\infty}$be a sequence of real numbers such that $x_1 = 1, x_2 = 1$ and $x_n = x_{n-1} + x_{n-2}$ for all $n \geq 3$. Show that
   $$\sum_{k=1}^{n} x_k = x_{n+2} - 1$$
   for all $n \geq 1$.
   (b) Let $\{y_n\}_{n=1}^{\infty}$be a sequence of real numbers $y_1 = y_2 = y_3 = 1$ and
   $$y_n = y_{n-1} + y_{n-2} + y_{n-3}$$
   for $n \geq 4$. Prove that $y_n < 2^n$ for all $n \geq 1$.

Extra Credit Puzzles:

1. You walk into Post office at R.V. College. Ms Samiha, Chief of Post, tells you that they have only Rs 3 and Rs 7 stamp. She also informs you that your envelope will need Rs 13 worth of stamps to make it to Indian Statistical Institute in Kolkata. Determine the combination (of Rs 3 and Rs 7) that you will purchase to get stamps worth Rs 13 ? Can you determine the set of all $n \geq 1$ such that Rs n worth of stamps can be created using stamps of Rs 3 and Rs 7.
2. Students in B.Math (hons.) first year just cannot handle Writing of Mathematics class. So n of them proceed to the football field, with chalk in hand, and stand there such that their mutual distances are all distinct. Then each of them throws the chalk at their nearest neighbour. Show that if n is odd then there is one person in the group who does not get hit by a chalk.

[1]**Office hours:** I will be in my office from 8:15-9am on Tue,Wed, Thu and from 2-3pm on Wed to answer any questions that you may have. Please feel free to drop by during these times to clarify any doubts that you may have.

Epsiloning Dancing Sequences and Generating Transcendental Numbers

1. Suppose $\{x_n\}_{n=1}^{\infty}$be a sequence of real numbers and let E its set of limit points. Decide if the following Show that $E = \{x\}$ for some $x \in \mathbb{R}$ if and only if $\{x_n\}_{n=1}^{\infty}$converges to x.

2. Let $\{x_n\}_{n=1}^{\infty}$be a sequence of real numbers. Suppose $I = \liminf_{n\to\infty} x_n$ and $S = \limsup_{n\to\infty} x_n$.
   (a) Supppose $S = I$ then show that $\lim_{n\to\infty} \mid x_n \mid = \mid I \mid = \mid S \mid$
   (b) Prove or disprove $\mid S \mid = \limsup_{n\to\infty} \mid x_n \mid$

3. Let $\{x_n\}_{n=1}^{\infty}$be a sequence of real numbers that is not bounded. Then show that there is either a subsequence that diverges to ∞ or a subsequence that diverges to $-\infty$.

4. In each of the cases below decide if $\{x_n\}_{n=1}^{\infty}$converges or not:
   (a) $x_n = \frac{2^n}{n!}$,
   (b) $x_n = \sqrt{n^2 - n} - n$
   (c) $x_n = nb^n$, for $b \in (0, 1)$.
   (d) $x_n = \frac{n^\alpha}{(1+p)^n}$ with $\alpha \in \mathbb{R}, p > 0$.

5. Consider $\{x_n\}_{n=1}^{\infty}$and $\{\mathrm{y}_n\}_{n=1}^{\infty}$given by

$$x_n = \left(1 + \frac{1}{n}\right)^n \quad \text{and} \quad y_n = \sum_{k=1}^{n} \frac{1}{k!}$$

   for all $n \in \mathbb{N}$. Show that
   (a) Using the comparison test appropriately show that $\{y_n\}_{n=1}^{\infty}$converges to a number between 2 and 3. *We call this limiting number e.*
   (b) Use the Binomail expansion to
      i. show that $x_n \leq y_n$ for all $n \geq 1$,
      ii. for all $m \geq 1$ $y_m \leq \liminf_{n\to\infty} x_n$ and
      iii. conclude that $\{x_n\}_{n=1}^{\infty}$also converges to e.
   (c) Show that $0 < e - s_n < \frac{1}{n!n}$ and conclude that e is not rational.
   (d) Is e an algebraic number ?

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