Homework 4
Due Date: September 5th, 2018

1. Let $S \subset \mathbb{R}$.
   - S is bounded above if there exists $M > 0$ such that $x \leq M$ for all $x \in S$. M is called an upper bound of S.
   - For a bounded set S, $\alpha \in \mathbb{R}$ is the least upper bound if
     - $x \in S$ then $x \leq \alpha$.
     - $\beta \in \mathbb{R}$ and $\beta < \alpha$ then there is a $x \in S$ such that $\beta < x$.

   (a) Express the statement S is not bounded above without using words of negation.

   (b) Decide which of the following mathematical statements are true (*with Justification*):

   i. 10.5 is the least upper bound of S and 9 is an upper bound of S.

   ii. Let S be finite set. If $\alpha \in \mathbb{R}$ is the least upper bound of S then $\alpha \in S$.

   iii. Let S be countable set and $\alpha \in \mathbb{R}$ be the least upper bound of S. Then for any $\beta \in \mathbb{R}$ such that $\beta < \alpha$ there is $x \in S$ such that $\beta < x < \alpha$.

   iv. Let $\alpha \in \mathbb{R}$ be the least upper bound of S. Then there exists a sequence $x_n \in S$ such that $\lim_{n\to\infty} x_n = \alpha$.

   v. Let $\alpha \in S$ be the least upper bound of S. Then $\alpha \in S$ is not an upper bound of $T := S \setminus \{\alpha\}$.

2. Let $f : \mathbb{R} \to \mathbb{R}$
   - f is bounded if there exists M such that $\mid f(x) \mid \leq M$ for all $x \in \mathbb{R}$.
   - f is increasing (or strictly increasing) if $f(x) < f(y)$ whenever $x < y$.
   - f is nondecreasing if $f(x) \leq f(y)$ whenever $x < y$.
   - f is decreasing (or strictly decreasing) if $f(x) > f(y)$ whenever $x < y$.
   - f is nonincreasing if $f(x) \geq f(y)$ whenever $x < y$.

   (a) Express the statement f is not bounded without using words of negation.

   (b) Express the statement f is not increasing without using words of negation.

   (c) Compare the definitions of nonincreasing and not increasing (the latter being the negation of increasing. Does one imply the other? Are there functions that satisfy one property, but not the other?

3. We say $\lim_{x\to 0} f(x) = 0$ if

   For every $\epsilon > 0$ there exists $\delta > 0$ such that $\mid f(x) \mid < \epsilon$ whenever $\mid x \mid < \delta$.

   Consider the following statements:

   (a) For every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in \mathbb{R}$, $\mid x \mid < \delta$ implies $\mid f(x) \mid < \epsilon$.

   (b) For every $\delta > 0$ there exists $\epsilon > 0$ such that for all $x \in \mathbb{R}$, $\mid x \mid < \delta$ implies $\mid f(x) \mid < \epsilon$.

   (c) There exists $\delta > 0$ such that for all $\epsilon > 0$ and for all $x \in \mathbb{R}$, $\mid x \mid < \delta$ implies $\mid f(x) \mid < \epsilon$.

   (d) For every $\epsilon > 0$ and for all $x \in \mathbb{R}$, there exists $\delta > 0$ such that $\mid x \mid < \delta$ implies $\mid f(x) \mid < \epsilon$.

   Decide which of the above versions are equivalent to the definition of

   $$\lim_{x\to 0} f(x) = 0$$

   and which are not. For those that are not equivalent to $\lim_{x\to 0} f(x) = 0$ determine, in as simple a language as possible, what they really define. Find examples (if they exist) of functions that satisfy the definition and of functions that don't satisfy it.