

$$\underline{P \Rightarrow Q}$$

DIRECT - Assume P , perform logical deductions,
conclude Q

Contra Positive

Assume $\neg Q$, perform logical
deductions, conclude $\neg P$

Contradiction

Assume $P \wedge \neg Q$, perform
logical deductions, arrive
at a contradiction

Q1 (columns 1, 2)

$$a \Rightarrow b$$

$$b \Rightarrow a$$

Q - group isom

$$\text{let } a, b \in G$$

$$a_1 \in G, a_2 \in G$$

$$\text{let } h \in G$$

Q1 columns 1, 2, 3

$a \Rightarrow b$ DIRECT

$b \Rightarrow a$ Contradiction

ϕ - group isomorphism

let $a, b \in G$ show $\phi(ab) = \phi(a)\phi(b)$

$g_1 \in G, g_2 \in G$ suppose $\phi(g_1) = \phi(g_2)$ then show $g_1 = g_2$

let $h \in G$ find $g \in G$ such that $\phi(g) = h$

columns 4, 5, 6

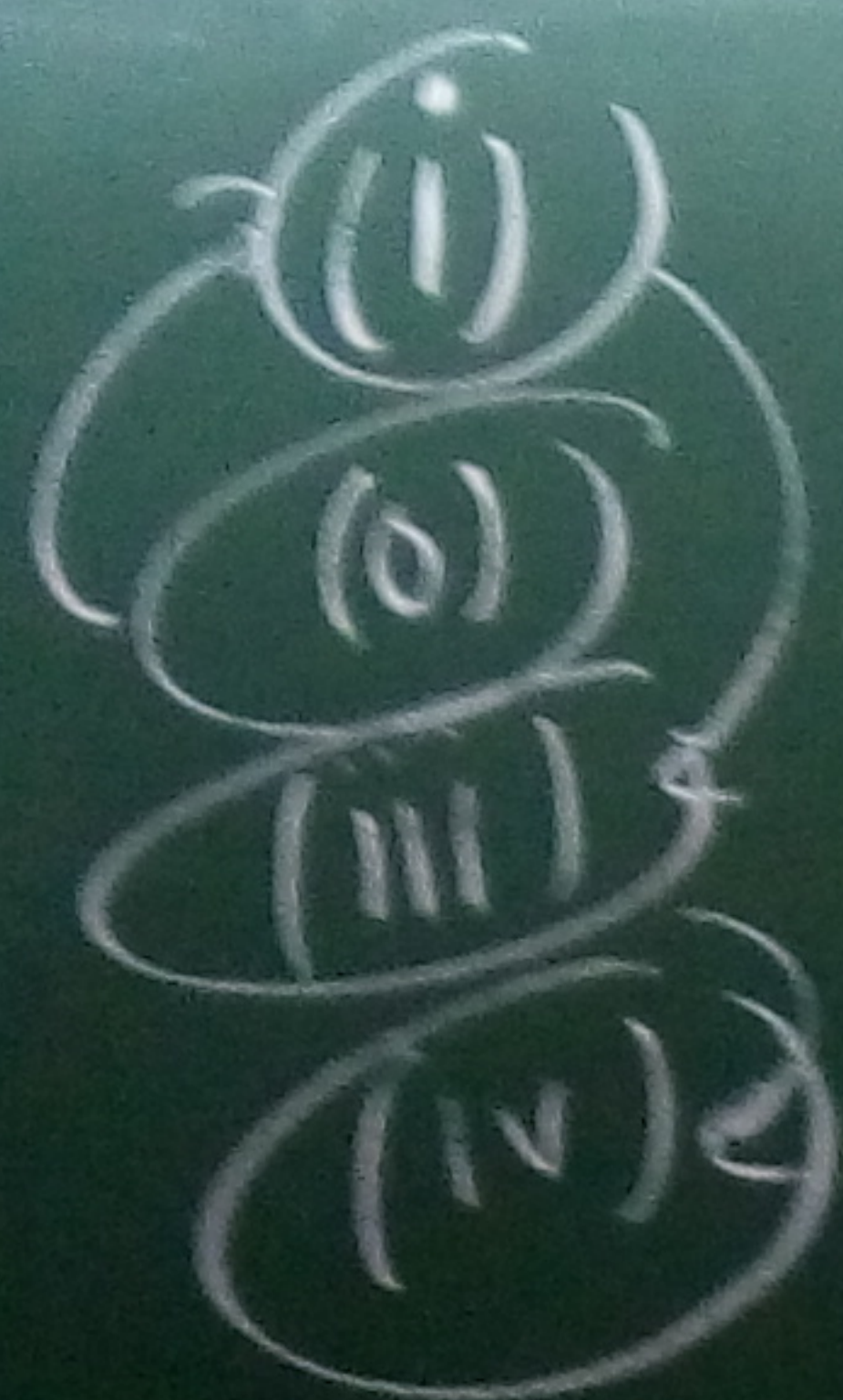
$a \Rightarrow b$ DIRECT

$b \Rightarrow a$ Contradiction

Q2



Q2



Definition.

Let $a_1, a_2, a_3, \dots$ be a sequence.

Let $f: \mathbb{N} \rightarrow \mathbb{N}$ be a strictly increasing function

Let $b: \mathbb{N} \rightarrow \mathbb{R}$ given by

$$b(k) = a_{f(k)}$$

we denote $b(k) = a_{n_k}$ with $n_k = f(k)$

$\mathbb{Z}$ in short one $\{a_{n_k}\}_{k \in \mathbb{Z}}$ and call it

the subsequence of $\{a_n\}_{n \in \mathbb{N}}$.

$$n_1 < n_2 < n_3 < \dots$$

(a) $\Rightarrow$ (c)

Assume (a) $\forall \epsilon > 0 \exists N \geq 1$ st $\forall n \geq N, |a_n| < \epsilon$.
— $\textcircled{*}$ —

let $\{a_{n_k}\}_{k \geq 1}$ be a sequence of $\{a_n\}_{n \geq 1}$

To show: there is a further subsequence of $\{a_{n_k}\}_{k \geq 1}$ that converges to 0.

Proof: let $f: \mathbb{N} \rightarrow \mathbb{N}$ be $f(k) = k$
 $b(k) = a_{f(k)} = a_{n_k} \quad \forall k \geq 1$

Claim: $a_{n_k} \rightarrow 0$ as $k \rightarrow \infty$ (Claim 5.6)

Let $\varepsilon_0 > 0$ be given.

From $(*)$, $\exists N_0 \geq 1$ st $\forall n \geq N_0$ $|a_n| < \varepsilon$ — $(**)$

Let $k \geq N_0$ $n_k \geq k \geq N_0$

by $(**)$ $\Rightarrow |a_{n_k}| < \varepsilon_0$

$\Rightarrow \exists N_0 \geq 1$ st $k \geq N_0$ $|a_{n_k}| < \varepsilon_0$

As $\varepsilon_0 > 0$ was arbitrary we have shown

$\forall \varepsilon > 0 \exists N \geq 1$ st
 $\forall k \geq N$ then $|a_{n_k}| < \varepsilon$

(iii)

(c)

$\forall$ subsequence $\{a_{n_k}\}_{k \geq 1}$ of the sequence $\{a_n\}_{n \geq 1}$
 $\exists$ a further subsequence of $\{a_{n_k}\}_{k \geq 1}$ denoted by
 $\{a_{n_{k_l}}\}_{l \geq 1}$ such that

$\forall \varepsilon > 0 \exists N \geq 1$ such that $l \geq N \Rightarrow |a_{n_{k_l}}| < \varepsilon$

$\neg (c)$

$\exists$ a subsequence $\{a_{n_k}\}_{k \geq 1}$ of the sequence $\{a_n\}_{n \geq 1}$ st

$\forall$ further subsequence of $\{a_{n_k}\}_{k \geq 1}$ denoted by $\{a_{n_{k_l}}\}_{l \geq 1}$

$\exists \varepsilon_0 > 0$ st $\forall N \geq 1 \exists l \geq N$ st $|a_{n_{k_l}}| \geq \varepsilon_0$

(iv) $(c) \Rightarrow (a)$ by contradiction

Assume $(c) \& \neg(a)$

$$\neg(a) \Rightarrow \exists \varepsilon_0 > 0 \quad \forall N \geq 1 \exists n \geq N \quad |a_n| \geq \varepsilon_0 \quad \text{--- } (*)$$

let $N=1$, $(*) \Rightarrow \exists n_1 \geq 1$ such that
 $|a_{n_1}| \geq \varepsilon_0$

let $N=n_1+1$, $(*) \Rightarrow \exists n_2 \geq n_1+1$ such that
 $|a_{n_2}| \geq \varepsilon_0$

Suppose $\{n_k\}$

let $N=n_k$

By induction

Suppose $\{n_i\}_{i=1}^k$ are given such that $n_i \geq n_{i-1} + 1 \quad \forall 2 \leq i \leq k+1$
 & $|a_{n_i}| \geq \epsilon_0 \quad \forall 1 \leq i \leq k$ for some $k \geq 1$

let $N = n_k + 1$ by (*) $\exists n_{k+1} > n_k + 1$ such that
 $|a_{n_{k+1}}| \geq \epsilon_0$

By induction there exists $f: \mathbb{N} \rightarrow \mathbb{N}$ strictly increasing
 such that $f(k) = n_k$ [defined above] $\forall k \geq 1$

$$\& \quad |a_{f(k)}| = |a_{n_k}| \geq \epsilon_0 \quad \forall k \geq 1$$

$k \geq 1$

$\Rightarrow \forall$ subsequence of $\{a_n\}_{k \geq 1}$, denoted by $\{a_{n_k}\}_{k \geq 1}$

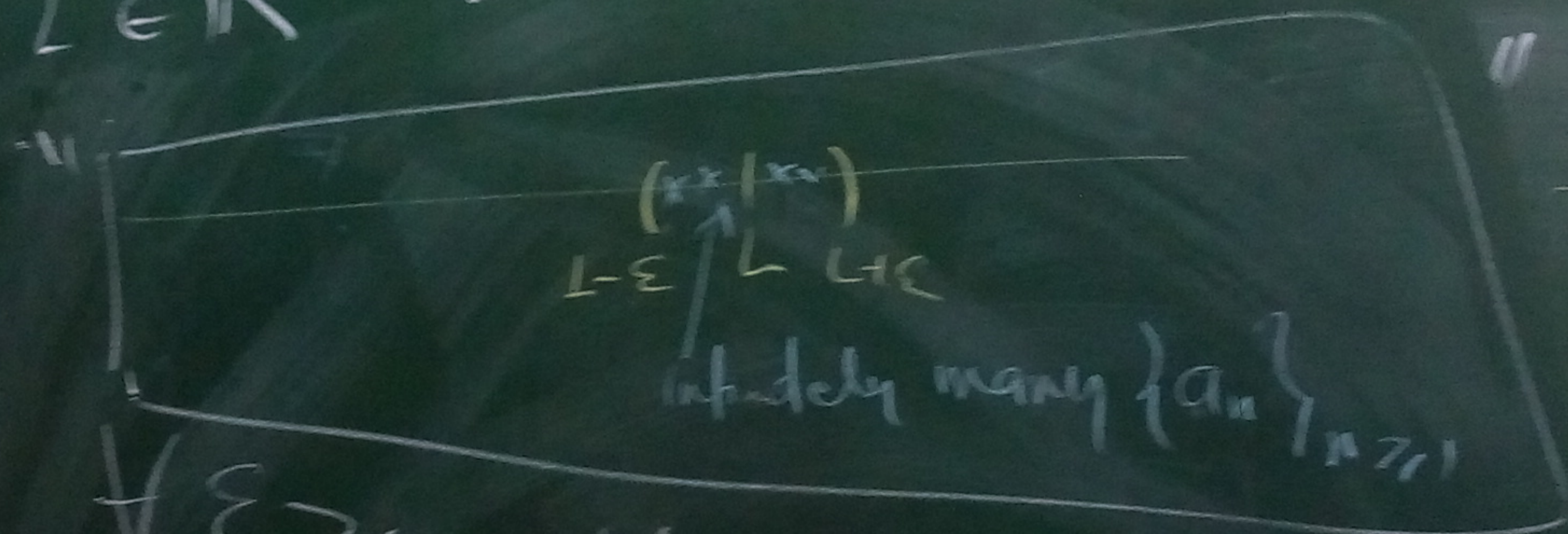
We have $|a_{n_k}| \geq \varepsilon_0 \quad \forall k \geq 1$

$\Rightarrow \neg (C)$

But we assumed (C) so we arrive at contradiction $\square$

Definition (Limit Point)

$L \in \mathbb{R}$ is a limit point of $\{a_n\}_{n \geq 1}$



if $\forall \varepsilon > 0 \quad \forall N > 1 \quad \exists n > N$

Such That $a_n \in (L - \varepsilon, L + \varepsilon)$

i.e. $|a_n - L| < \varepsilon$

(a) $\Rightarrow$ (b)

$\exists M > 0$ such that $|a_n| \leq M \quad \forall n \geq 1$ (Bounded)

$\stackrel{\text{ex}}{\Rightarrow}$ By definition $A \subseteq [-M, M]$

$\Rightarrow \sup(A) = L \in \mathbb{R}$

To show (b)

$\varepsilon_0 > 0$ be given

$\exists N > 1$ st $\forall n > N \quad a_n < L + \varepsilon_0$
 $\forall M > 1 \quad \exists N > M$ st $a_n > L - \varepsilon_0$

$L + \frac{\epsilon_0}{2}$ is NOT a limit point of $\{a_n\}_{n \geq 1}$
 $\exists \delta > 0 \quad \exists N > 1 \quad \text{st} \quad \forall n \geq N$

$$|a_n - (L + \frac{\epsilon_0}{2})| > \delta$$

ie $L - \frac{\epsilon_0}{2} - \delta < a_n < L + \frac{\epsilon_0}{2} + \delta$

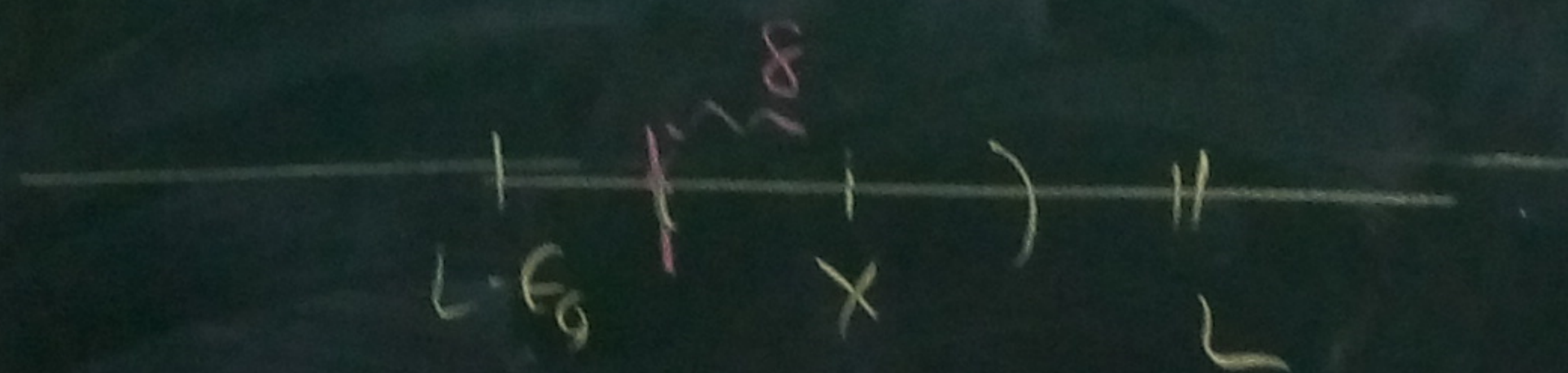
$$\delta_0 = \min(\frac{\epsilon_0}{2}, \delta) \Rightarrow \exists N > 1 \quad \text{st} \quad \forall n \geq N$$

$$a_n < L + \frac{\epsilon_0}{2} + \delta < L + \frac{\epsilon_0}{2} + \frac{\epsilon_0}{2} = L + \epsilon_0$$

To show
①

absence $(L - \epsilon_0)$ is NOT an upper
bound of A .

$\exists x \in A$ such that $L - \epsilon_0 < x \leq L$



$$\delta = \frac{x - (L - \epsilon_0)}{2}$$

By definition limit-point, $\forall N \geq 1 \exists n \geq N$
such that $a_n > x - \delta > L - \epsilon_0$

By choice of
 δ

$L - \epsilon_0$

As $\varepsilon_0 > 0$ was arbitrary we are done.

$\exists M > 0$ such that $|a_n| \leq M \quad \forall n \geq 1$ (Bounded)

$\Rightarrow$

By definition

$$A \subseteq [-M, M]$$

Assume (a)

$\Rightarrow$

$$\sup(A) = L \in \mathbb{R}$$

To show (b)

$\varepsilon_0 > 0$ be given

$\exists N > 1$ st $\forall n > N$

$$a_n < L + \varepsilon_0$$

(+) $\forall M > 1 \exists n > M$ st

$$a_n > L - \varepsilon_0$$

L is NOT a limit point of $\{a_n\}_{n \geq 1}$
 $\exists \delta > 0$ & $\exists N \geq 1$ st $\forall n \geq N$

$$|a_n - (L + \frac{\epsilon_0}{2})| > \delta$$

ie $L - \frac{\epsilon_0}{2} - \delta < a_n < L + \frac{\epsilon_0}{2} + \delta$

$$\delta_0 = \min(\frac{\epsilon_0}{2}, \delta) \Rightarrow \exists N \geq 1 \text{ st } \forall n \geq N$$

$$a_n < L + \frac{\epsilon_0}{2} + \delta_0 < L + \frac{\epsilon_0}{2} + \frac{\epsilon_0}{2} = L + \epsilon_0$$