Recall

$$f(x,y) = ax^2 + bxy + cy^2$$

$$= a\left(x + \frac{by}{2a}\right)^2 + \left(-\frac{b^2}{4a} + c\right)y^2$$

$$= \frac{1}{4a}\left[4a^2\left(x + \frac{by}{2a}\right)^2 + (4ac - b^2)y^2\right]$$

$$4ac - b^2 < 0$$

$(a \neq 0)$ $$f(x,y) = \underbrace{*(\ )^2}_{\geq 0} - \underbrace{*(\ )^2}_{\geq 0}$$

"Convinc

u

that

$4ac$

$a < 0$

$a > 0$

"Convince by change of coordinates

$u = x + \frac{b}{2a} y$ & $v = y$" ~ $u^2 - v^2 = g(u,v)$

that we will get saddle point at $(0,0)$

$4ac - b^2 > 0$

$f(x,y) = \frac{1}{4a} \Big[\underbrace{*(\ \ )^2}_{\geq 0} + \underbrace{*(\ \ )^2}_{\geq 0} \Big]$

$f(x,y) = 0$

$\iff x + \frac{b}{2a} y = 0$ & $y = 0$

$\iff x = 0$ & $y = 0$

$a < 0$ — local maximum $\equiv f(x,y) \leq 0$ & $(0,0)$ is local maximum

$a > 0$ — local minimum $\equiv f(x,y) \geq 0$ & $(0,0)$ is local minimum

$4ac$ -

• Think

≡ along

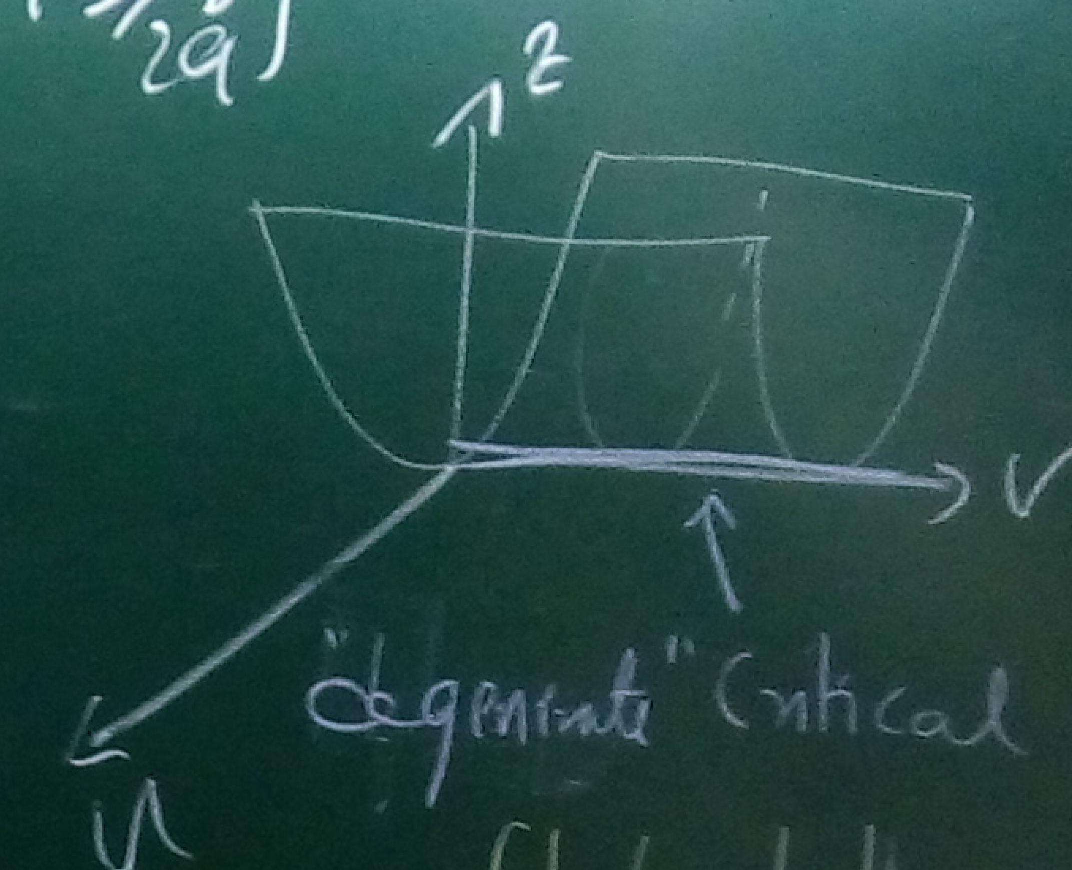
≡ In g

Second De

• Let (

$4ac - b^2 = 0$. $\quad f(x,y) = \frac{1}{}\, a\left(x + \frac{b}{2a}y\right)^2$

- Think of $g(u,v) = u^2$



$\equiv$ along one direction no change occurs

"degenerate" critical points

$\equiv$ In general! cannot say local min/max or saddle point – flat bottom valley

Second Derivative test for $f : \mathbb{R}^2 \to \mathbb{R}$

- Let $(\alpha, \beta) \in \mathbb{R}^2$ be a critical point.

$$f_x(\alpha,\beta) = 0 = f_y(\alpha,\beta).$$

"nice"

- $A = f_{xx}(\alpha, \beta)$ $\quad B = f_{xy}(\alpha, \beta) \equiv f_{yx}(\alpha, \beta)$

  $C = f_{yy}(\alpha, \beta)$

Second Derivative test

- $AC - B^2 > 0$ & $\begin{cases} A > 0 & \text{local minimum} \\ A < 0 & \text{local maximum} \end{cases}$
- $AC - B^2 < 0$ — Saddle point
- $AC - B^2 = 0$ — Degenerate critical point

Spe

$u = at^2 -$

$b^2 - 4ac$

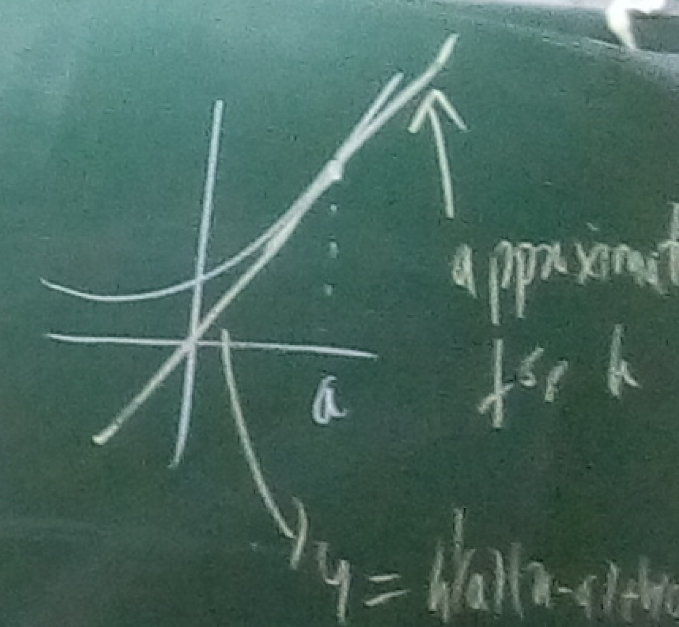
$b^2 - 4a$

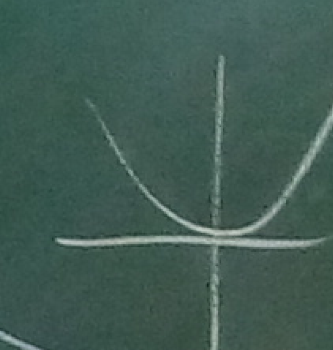
1-variable

$h : \mathbb{R} \to \mathbb{R}$ $\qquad$ $h'(a)$, $h''(a)$ exists

x "near" a $\Rightarrow$ $h(x) \approx h(a) + (x-a)h'(a)$

$\boxed{h'(a) = 0}$ $\Rightarrow$ $h(x) \approx h(a) + \frac{1}{2}(x-a)^2 h''(a)$





Worksheet in August

"$h(x) = x^5$"

2-variable

$f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) \approx f(\alpha,\beta) + (x-\alpha) f_x(\alpha) + (y-\beta) f_y(\beta)$$

$\boxed{f_x(\alpha,\beta) = 0 = f_y(\alpha,\beta)}$

$$f(x,y) \approx f(\alpha,\beta) + \frac{1}{2}(x-\alpha)^2 f_{xx}(\alpha,\beta) + \frac{1}{2}(y-\beta)^2 f_{yy}(\alpha,\beta) + (x-\alpha)(y-\beta) f_{xy}(\alpha,\beta)$$

Special case: $f(x,y) = ax^2 + bxy + cy^2 + \text{"}\frac{1}{10^7}x^5\text{"}$

- $A \equiv f_{xx}(\alpha,\beta) = 2a$, $B \equiv f_{xy}(\alpha,\beta) = b$, $C \equiv f_{yy}(\alpha,\beta) = 2c$

$$AC - B^2 = 4ac - b^2$$

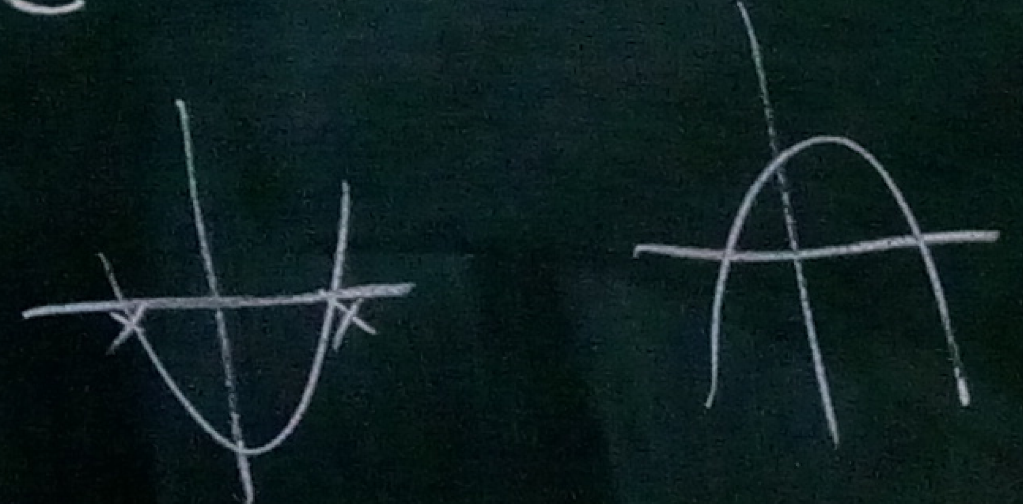
- $f(x,y) = y^2\left[a\left(\frac{x}{y}\right)^2 + b\left(\frac{x}{y}\right) + c\right]$

$a_n \le b_n$

$\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n$

$u = at^2 + bt + c$

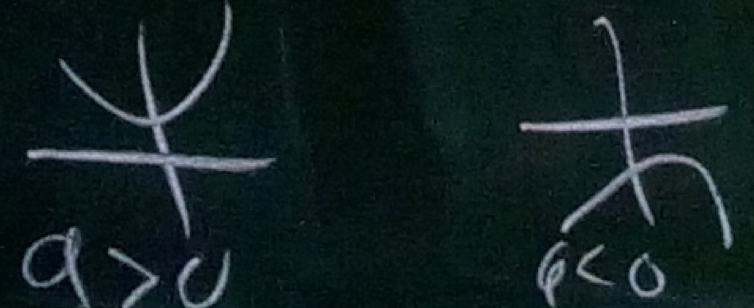
$b^2 - 4ac > 0$



— Saddle point

$b^2 - 4ac = 0$

$b^2 - 4ac < 0$



— local min/max