Elementary Inequalities.

$\boxed{xx}$ Square root of positive number are positive

$$a < b \implies ac < bc \quad (*)$$
iff $c > 0$

Assume: $a, b \in \mathbb{R}$
$c > 0$ & $c \in \mathbb{R}$

[Ordered: -
$a, b \in \mathbb{R} \Rightarrow a = b$ or $a < b$ or $a > b$
(?) $z \in \mathbb{R}$
$\rightarrow z > 0$ & $x < y$ then $x + z < y + z$
$\rightarrow x > 0$ & $y > 0$ then $xy > 0$]

Result 1: $0 < a < b$ then $a^2 < ab < b^2$
and $0 < \sqrt{a} < \sqrt{b}$

Pros

Proof

$a \cdot a = a^2,\ b a = ab$

- $a < b$ and $a > 0$, so (*) $\Rightarrow$ $a \cdot a < b \cdot a \Rightarrow a^2 < ab$.

  $a < b$ and $b > 0$, so (*) $\Rightarrow$ $a \cdot b < b b \Rightarrow ab < b^2$

$$\Rightarrow \quad a^2 < ab < b^2$$

- Assume $\sqrt{a} \geq \sqrt{b}$. As $\sqrt{b} > 0$ by (**) we will have

$$0 < \sqrt{b} \leq \sqrt{a}$$

By the previous part applied to $\sqrt{b}$ & $\sqrt{a}$ (& by its proof) we have $b \leq \sqrt{b}\sqrt{a} < a$

but this is a contradiction as $a < b$ by assumption. □

Definition 2: $x \in \mathbb{R}$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases}$$

$$\Longrightarrow \begin{cases} \forall z \in \mathbb{R} \\ \cdot\ z \leq |z| \\ \cdot\ |xy| = |x||y| \quad \forall x, y \in \mathbb{R} \\ \cdot\ |x|^2 = x^2 \end{cases}$$

Result 3: - (Triangle Inequality) $x, y \in \mathbb{R}$ Then

$$|x+y| \leq |x| + |y|.$$

Proof: - We always have $2xy \leq 2|x||y|$

$$\boxed{\begin{aligned} \cdot\ 2xy &\leq |2xy| \\ &= |2x|\,|y| \\ &= |2|\,|x||y| \\ &= 2|x||y| \end{aligned}}$$

$x^2 > 0$

$y^2 > 0$

$$\Rightarrow x^2 + y^2 + 2xy \leq x^2 + y^2 + 2|x||y|$$

$$\Rightarrow x^2 + y^2 + 2xy \leq |x|^2 + |y|^2 + 2|x| \cdot |y|$$

$$\Rightarrow (x+y)^2 \leq (|x| + |y|)^2 \Rightarrow |x+y|^2 \leq (|x| + |y|)^2 \Rightarrow |x+y| \leq |x| + |y|$$

Result 1 to
$a = |x+y|^2$ & $b = (|x|+|y|)^2$