Recall:

$\mathrm{Card}(X) \leq \mathrm{Card}(Y)$ if there is $f: X \to Y$ and f is an injection

$\mathrm{Card}(X) = \mathrm{Card}(Y)$ if there is $f: X \to Y$ and f is a bijection

$\mathrm{Card}(X) \geq \mathrm{Card}(Y)$ if there is $f: X \to Y$ and f is a surjection

R4 If $\mathrm{Card}(X) \leq \mathrm{Card}(Y)$ & $\mathrm{Card}(Y) \leq \mathrm{Card}(X)$ then $\mathrm{Card}(X) = \mathrm{Card}(Y)$

Proof

$f: X \to$

$g: Y \to$

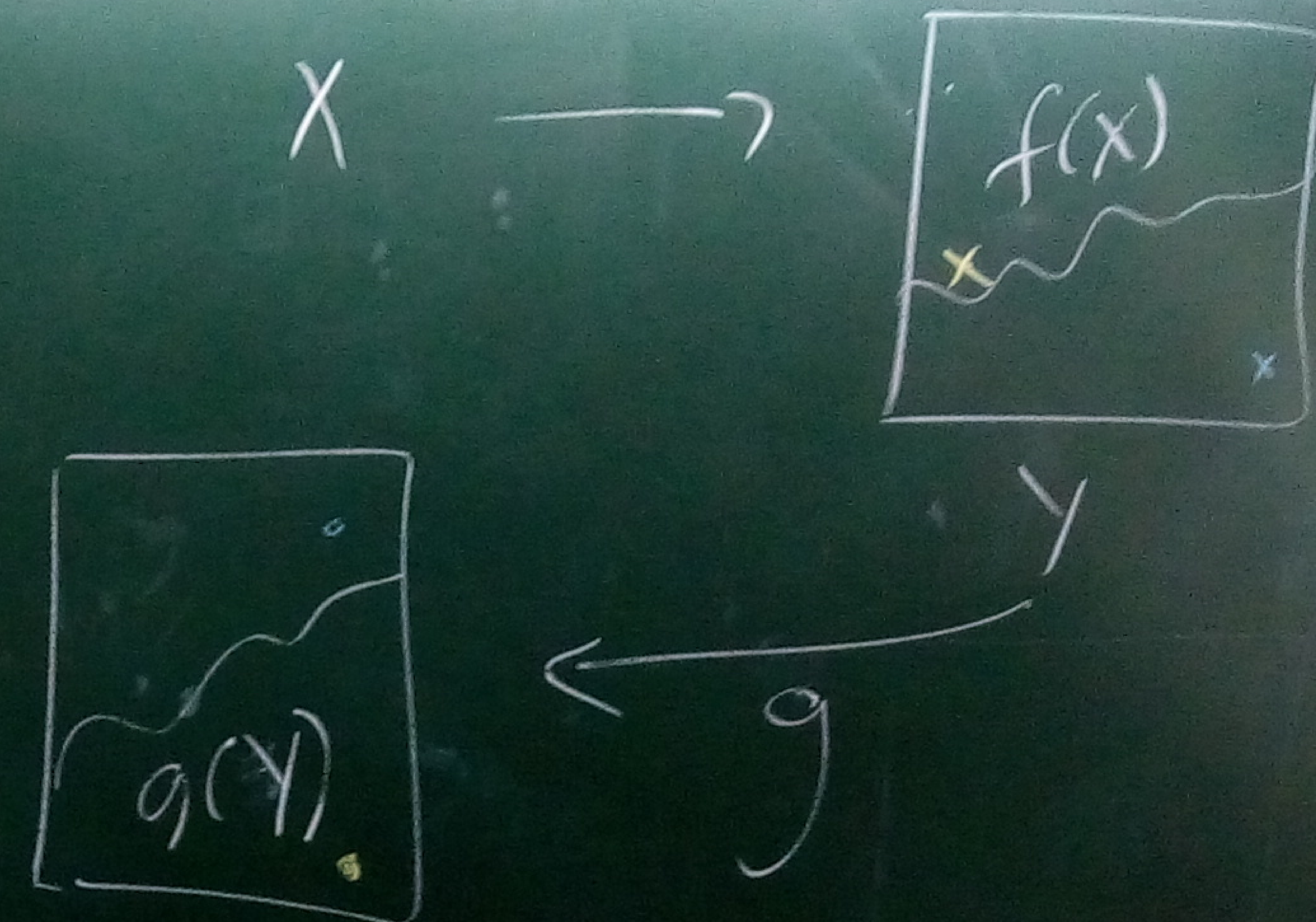
$x_0 \in X$

3 Possibili

Proof

$f: X \to Y$ injection

$g: Y \to X$ injection



$x_0 \in X$ for $k \geq 1$, if $x_{k-1} \in g(Y)$ then $x_k = g^{-1}(x_{k-1}) \in Y$

if $x_k \in f(X)$ then $x_{k+1} = f^{-1}(x_k) \in X$

3 Possibilities: (1) generate $\{x_k\}_{k \geq 1}$ starting from $x_0 \equiv x_0 \in X$

or $\underline{\underline{(2)}}$ $\exists N$: $x_N \notin g(Y) \equiv x_0 \in X_x$

$\underline{\underline{(3)}}$ $\exists N$ $x_N \notin f(X) \equiv x_0 \in X_y$

$$X = X_\infty \sqcup X_x \sqcup X_y$$

Same procedure, reverse roles of f & g:

$$Y = Y_\infty \sqcup Y_y \sqcup Y_x$$

$f: X_\infty \to Y_\infty$
$f: X_x \to Y_x$
$g: Y_y \to X_y$
are all surjections

Take $h: X \to Y$

$$h(x) = \begin{cases} f(x) & x \in X_\infty \cup X_x \\ g^{-1}(x) & x \in X_y \end{cases} \quad \square$$

Bijection!

$x_n \neq x_m \quad \forall n \neq m \quad n, m \in \mathbb{N}$

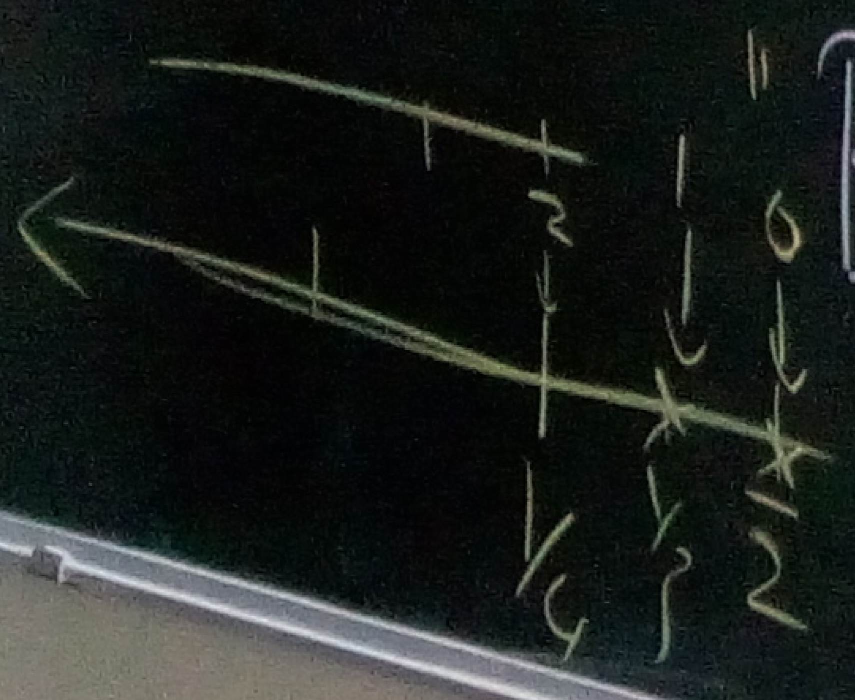
Recall: $\{x_n\}_{n \geq 1}$ $\quad A = \{x_n \mid n \geq 1\} \qquad B = \{x_n \mid n \geq 2\}$

Ex $f : A \to B$ $\quad$ 1-1 & onto

$$f(x_n) = x_{n+1}$$

$h : [0,1] \to (0,1)$

$$A = \{0,1\} \cup \left\{\frac{1}{k} : k \geq 2\right\} \subseteq [0,1]$$



$$B = \left\{\frac{1}{k} : k \geq 1\right\} \subseteq (0,1)$$

$f: A \to B$

$f(0) = \frac{1}{2}$

$f(1) = \frac{1}{3}$

$k \geq 2 \quad f\left(\frac{1}{k}\right) = \frac{1}{k+2}$

$h: [0,1] \to (0,1)$

$$h(x) = \begin{cases} f(x) & x \in A \\ x & x \notin A \end{cases}$$

h (=) bijection

$A = \mathbb{N} \quad B = \mathbb{R} \to \text{card}(\mathbb{N}) < \text{card}(\mathbb{R})$

1. $\text{card}(\mathbb{N}) < \text{card}(P(\mathbb{N}))$
2. $\text{card}(P(\mathbb{N})) = \text{card}(\mathbb{R})$

Pf 1 $\quad f: \mathbb{N} \to P(\mathbb{N})$

$\boxed{1-1} \quad f(n) = \{n\}$

Any $g: \mathbb{N} \to P(\mathbb{N})$ cannot be surjection

R)

$A = \{ n \in \mathbb{N} \mid n \notin g(n) \}$ Ex: $A \in \mathcal{P}(\mathbb{N})$, $A \neq g(k)$ for any $k \in \mathbb{N}$ $\square$

Proof of ②

$f : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ need to be 1-1 onto

$$f(A) = \begin{cases} \sum_{n \in A} \frac{1}{2^n} & \mathbb{N} \setminus A \text{ is infinite} \\ 1 + \sum_{n \in A} \frac{1}{2^n} & \mathbb{N} \setminus A \text{ is finite} \end{cases}$$

$\boxed{f \text{ is } 1\text{-}1}$

$\text{card}(\mathcal{P}(\mathbb{N})) \leq \text{card}(\mathbb{R})$

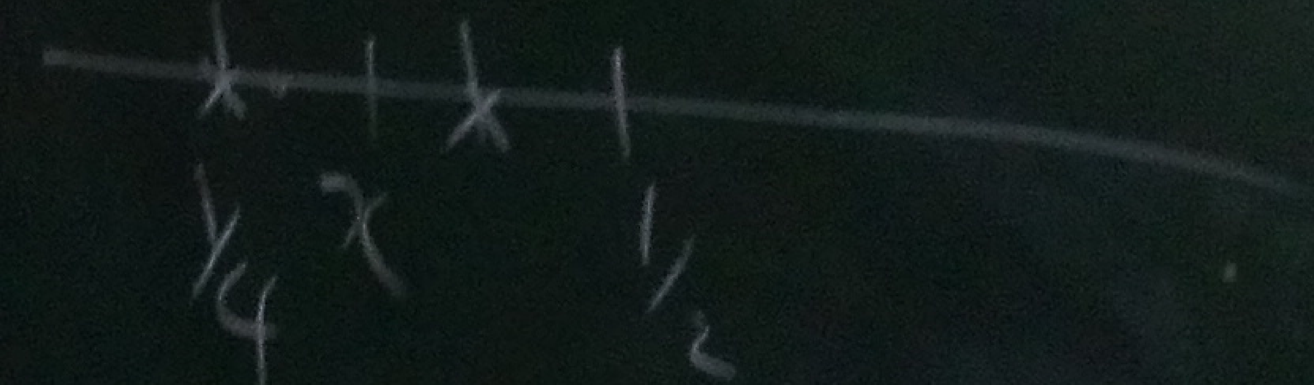
$A = \emptyset$

$$g: \mathcal{P}(\mathbb{N}) \longrightarrow \mathbb{R}$$

$$g(A) = \sum_{n \in A} \frac{1}{2^n} \in [0,1)$$

$$g: \mathcal{P}(\mathbb{N}) \to [0,1)$$

is indeed a surjection



$f: A \to B$

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$h: [0,1] \to (0,1)$

$$h(x) = \begin{cases} f(x) & x \in A \\ x & x \notin A \end{cases}$$

h (→) + ... te

$A = \mathbb{N} \qquad B = \mathbb{R} \to \mathrm{Card}(\mathbb{N}) < \mathrm{Card}(\mathbb{R})$

1. $\mathrm{Card}(\mathbb{N}) < \mathrm{Card}(P(\mathbb{N}))$

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Pf 1

$\boxed{1-1}$

$f: \mathbb{N} \to P(\mathbb{N})$

$f(n) = \{n\}$

Any $g: \mathbb{N} \to P(\mathbb{N})$

cannot be surjection

Proof

(R)

$A = \{ n \in \mathbb{N} \mid n \notin g(n) \}$

Ex: $A \in \mathcal{P}(\mathbb{N})$, $A \neq g(k)$ for any $k \in \mathbb{N}$

$\square$

Proof of ②

$f : \mathcal{P}(\mathbb{N}) \to \mathbb{R}$ need to be 1-1 onto

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$\text{Card}(\mathcal{P}(\mathbb{N})) \leq \text{Card}(\mathbb{R})$