Recall (Inclusion - Exclusion)

U - universe set $|U| < \infty$

$\{A_i\}_{1 \le i \le n}$ $A_i \subseteq U$

N_ϕ = # of elements which do not belong to any of A_i, $1 \le i \le n$

$$N_\phi = \sum_{S \subseteq [n]} (-1)^{|S|} \left| \bigcap_{i \in S} A_i \right|$$

Q3 (General Version)

$$S_n \equiv \{ \sigma : [n] \to [n] \mid \sigma \text{ is 1-1 and onto} \}$$

$$n! = |S_n|$$

Derangement : $\sigma \in S_n$ such that $\sigma(i) \neq i$ for any $1 \leq i \leq n$

EQUALLY LIKELY

Sample space - S_n

Event :- All subsets of S_n

$\mathbb{P} : S_n \to [0,1]$

$$P(A) = \frac{|A|}{|S_n|} = \frac{|A|}{n!}$$

$\sigma \in S_n$ $P(\{\sigma\}) = \frac{1}{n!}$

$$P(A) = \sum_{\sigma \in A} P(\{\sigma\})$$

<u>Question</u>: $\mathbb{P}(D_n) = ?$

$$P(D_n) = \frac{|D_n|}{n!}$$

$$A_i = \{\sigma \in S_n \mid \sigma(i) = i\} \quad \text{for } 1 \le i \le n$$

$D_n \equiv \quad \sigma \in S_n$ which do not belong to any of A_i $1 \le i \le n$

$$|D_n| = \sum_{i \in S} (-1)^{|S|} \left| \bigcap_{i \in S} A_i \right|$$

<u>Question</u>: $\mathbb{P}(D_n) = ?$

$$P(D_n) = \frac{|D_n|}{n!}$$

$A_i = \{ \sigma \in S_n \mid \sigma(i) = i \}$ for $1 \le i \le n$

$D_n \equiv$ $\sigma \in S_n$ which do not belong to any of A_i

$$|D_n| = \sum_{S \subseteq [n]} (-1)^{|S|} \left| \bigcap_{i \in S} A_i \right|$$

$k \geq 1 \quad |S| = k \quad \sigma \in \bigcap_{i \in S} A_i$

$$\left|\bigcap_{i \in S} A_i\right| = (n-k)!$$

$$1 \ldots k \quad \underbrace{k+1 \ldots n}_{(n-k)\text{ elements}}$$

k

There are ${}^{n}C_{k}$ such sets of size k

$$|D_n| = \sum_{S \subseteq [n]} (-1)^{|S|} \left|\bigcap_{i \in S} A_i\right|$$

$$= \sum_{S \subseteq [n]} \sum_{\substack{|S|=k \\ k=0}}^{n} (-1)^{|S|} \left| \bigcap_{i \in S} A_i \right|$$

$$= \sum_{k=0}^{n} (-1)^k (n-k)! \left[\sum_{S \subseteq [n]} 1(|S|=k) \right]$$

$$= \sum_{k=0}^{n} (-1)^k (n-k)! \; {}^{n}C_k$$

$$= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

$$P(D_n) = \frac{n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}}{n!}$$

$$\Rightarrow \mathbb{P}(D_n) = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

As $e^{-1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!}$, $P(D_n) \approx e^{-1}$