Recall.

$\mathbb{N}$ - $\boxed{\text{Peano's Axioms}}$

$\to$ number system (Romans)

$\boxed{\text{Well ordering}}$

least element

$\mathbb{N} \longrightarrow$ Induction hypothesis

HW ideas

Number theoretic facts

$\mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow (\mathbb{R}, +, \cdot)$

(complete ordered field

X, Y b

Expressi

Card (X)

Card (X

Card (x

"Cardinality

Read on

X, Y be two non-empty sets we define

Expressions

$\mathrm{card}(X) \leq \mathrm{card}(Y)$ if there is $f: X \to Y$ such that f is 1-1

$\mathrm{card}(X) = \mathrm{card}(Y)$ if there is $f: X \to Y$ such that f is 1-1 & onto

$\mathrm{card}(X) \geq \mathrm{card}(Y)$ if there is $f: X \to Y$ such that f is onto

"Cardinality of X"
Read as:

- "card (X)" has no relevance alone. (Except in finite regime)
- Forget ϕ. Convention $\mathrm{card}(\phi) < \mathrm{card}(X)$
$\mathrm{card}(X) > \mathrm{card}(\phi)$

- Assume unless stated X, Y are non-empty

Result 1. $\text{Card}(X) \leq \text{Card}(Y)$ iff $\text{Card}(Y) \geq \text{Card}(X)$

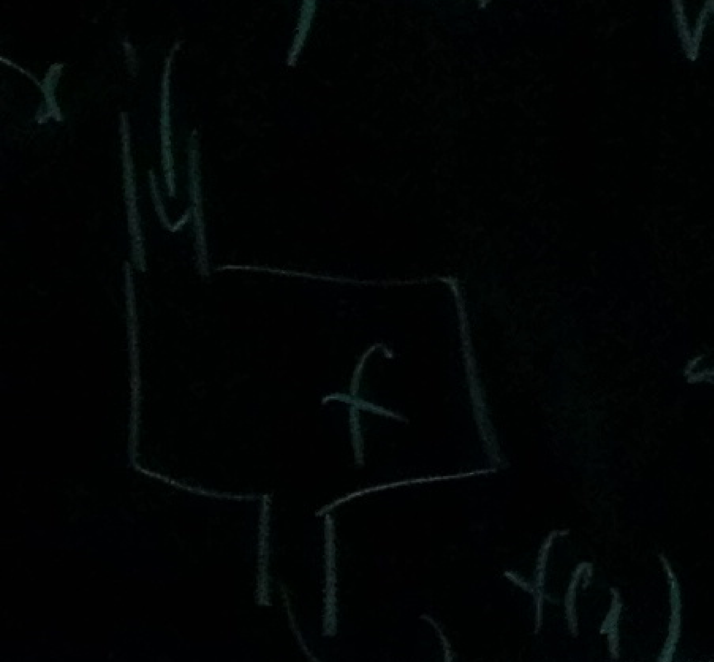
Proof. Let $\text{Card}(X) \leq \text{Card}(Y)$

There is $f: X \to Y$ such that f is an injection

$$g: Y \to X$$

Let $x_0 \in X$. $g(y) = \begin{cases} f^{-1}(y) & y \in f(X) \\ x_0 & y \notin f(X) \end{cases}$

Ex: g is well defined function



? Ex

Ex: g is onto. $\Rightarrow \text{Card}(Y) \geq \text{Card}(X)$

Let $\text{Card}(Y) \geq \text{Card}(X)$

There is $g: Y \to X$ such that g is surjection

$g^{-1}(x) = \{ y \in Y \mid g(y) = x \}$

$\boxed{f \in \prod_{x \in X} g^{-1}(x)}$

$f: X \to \bigcup_{x \in X} g^{-1}(x)$

$x_0 \longrightarrow$ any element in $g^{-1}(x_0)$ fixed

- $g^{-1}(x) \cap g^{-1}(y) = \phi$ if $x = y$
- f is 1-1

$\mathrm{card}(Y) \leq \mathrm{card}(X)$

R2: $\mathrm{card}(X) \leq \mathrm{card}(Y)$ or $\mathrm{card}(Y) \leq \mathrm{card}(X)$

Proof: $\mathcal{I} = \{ f : A \to Y \,/\, A \subseteq X \text{ \& } f \text{ is } 1\text{-}1 \}$

"Think" of $\mathcal{I} \subseteq X \times Y$ — "{ Definition of $f : X \to Y$ }"

$(\mathcal{I}, \subseteq)$

partial order on $\mathcal{I}$

ie $A, B \in \mathcal{I}$

$A \subseteq B$ & $B \subseteq A \Rightarrow A = B$

$A \subseteq A$

$A \subseteq B$ & $B \subseteq C \Rightarrow A \subseteq C$

linear order ✗

$A, B \in \mathcal{I}$ $A \subseteq B$ or $B \subseteq A$

E.g.

X maximal element of $\mathcal{I}$ $\quad$ $x \in \mathcal{I}$ is a maximal element

y $\quad$ $x \leq y$ & $y \in \mathcal{I} \Rightarrow y = x$

Zorn's lemma: X is a partially ordered set &

every linearly ordered subset of X has a maximal element. (upper bound). Then X has a maximal element.

E.g. $X = \left\{ A_\alpha \in \mathcal{I} \;\middle|\; \begin{array}{ll} \alpha \in J & A_\alpha \subseteq A_{\alpha'} \\ \alpha' \in J & \text{or } A_{\alpha'} \subseteq A_\alpha \end{array} \right\} \overset{\text{Zorn}}{\Longrightarrow}$ $\mathcal{I}$ has a maximal element

$\bigcup_{\alpha \in J} A_\alpha$ is maximal element of Y

Let $f \in \mathcal{I}$ be a maximal element. $f : A \to Y$ where

$A \subseteq X$

$B = \{ y \in Y \mid \text{there is an } x \in X \text{ s.t. } f(x) = y \}$.

$f : A \to B$ is surjective

Claim: $A = X$ or $B = Y$

$A = X \Rightarrow \operatorname{card}(X) \leq \operatorname{card}(Y)$

$B = Y \qquad \operatorname{card}(Y) \leq \operatorname{card}(X)$

Suppose $A \neq X$, $B \neq Y$

$\exists\ x_0 \in X \quad x_0 \notin A$

$y_0 \in Y \quad y_0 \notin B$

$\tilde{f}(x_0) = y_0$

$\tilde{f}(x) = f(x) \quad \forall x \in A$

$\tilde{f} : A \cup \{x_0\} \to B \cup \{y_0\}$

$A \cup \{x_0\} \times B \cup \{y_0\} \supsetneq A \times B$

$A \times B$ is maximal & have contradiction

I - a set

$$\alpha \in I \longrightarrow X_\alpha$$

$\boxed{\text{Axiom of choice}}$

$$\text{"}\prod_{\alpha \in I} X_\alpha\text{"} = \left\{ f : I \longrightarrow \bigcup_{\alpha \in I} X_\alpha \;\&\; f(\alpha) \in X_\alpha \right\}$$

can choose $f \in \prod_{\alpha \in I} X_\alpha$

$$\mathbb{R}^n = \{ (x_1, \dots, x_n) \mid x_i \in \mathbb{R} \}$$

$$= \left\{ f : \{1, \dots, n\} \to \bigcup_{i=1}^{n} R_i \;,\; R_i = \mathbb{R} \right\}$$