$k \geq 1, n \geq 1$

Pigeon Hole Principle

If we have to place more than kn objects into n boxes then one box will have to contain more than k objects.

Proof.- (By contradiction) Suppose we

pla

As

By

$\Rightarrow$

Th

$\Rightarrow$

place m_i objects in box i for $1 \le i \le n$.

Assume $m_i \le k \quad \forall\, 1 \le i \le k$

By induction – $\sum_{i=1}^{n} m_i \le nk = kn$

$\Rightarrow$ Total number of objects placed is less than or equal to kn.

This contradicts the hypothesis

$\Rightarrow \exists\, i \;\; 1 \le i \le n$ such that $m_i > k$. $\square$

1 [Sketch]

Suppose not, consider $ax \not\equiv 1 \bmod p \;\; \forall x \in$

$a, 2a, 3a, \ldots, (p-1)a$

$\boxed{PH}$

$\exists\, i, j$

$ai \equiv aj \bmod p$

$\Rightarrow p \mid a(i-j)$

$p \nmid a \Rightarrow p \mid (i-j)$

$i, j < p \Rightarrow i-j = 0$

Contradiction as $i \ne j$

$\langle 1, 2, \ldots, p-1 \rangle$

$p-1$ numbers
$p-2$ classes

② [Sketch]

$\mathbb{Z} \times \mathbb{Z}$

(o, e)
(e, e)
(o, o)
(e, o)

4 classes

5 points $\Rightarrow$ (PH)

2 in one class

$\Rightarrow$ midpoint of atleast one pair is an integer. □

□

Inclusion-Exclusion Principle

U - universe set $|U| < \infty$

$n \geq 1$ Let A_i, $1 \leq i \leq n$ be subsets of U

$N(\phi) \equiv$ number of points in U that do not belong to any A_i.

$$N(\phi) = \sum_{S \subseteq [n]} (-1)^{|S|} \left| \bigcap_{i \in S} A_i \right|$$

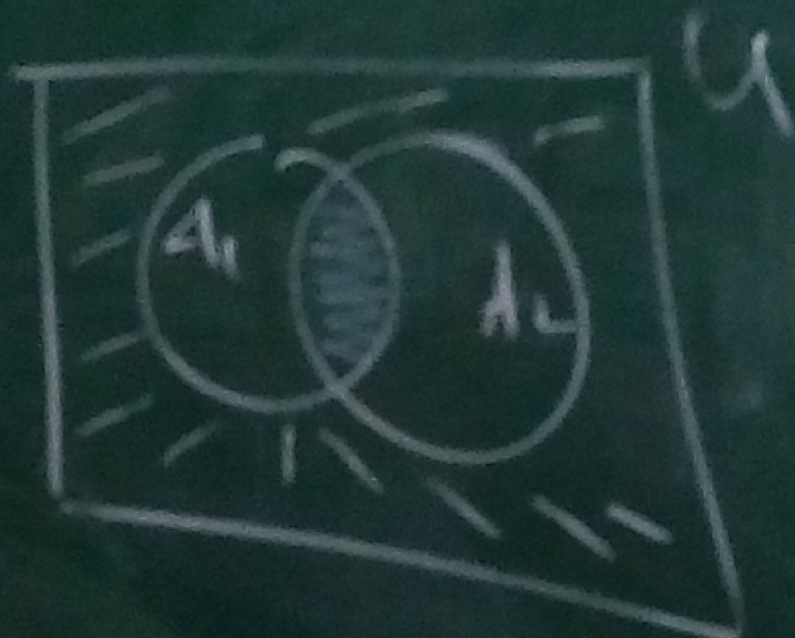
$[n] = \{1, \ldots, n\}$.

Proof:

$n=1$ $\quad A_1 \subseteq U$

$$N(\phi) = |U| - |A_1|$$

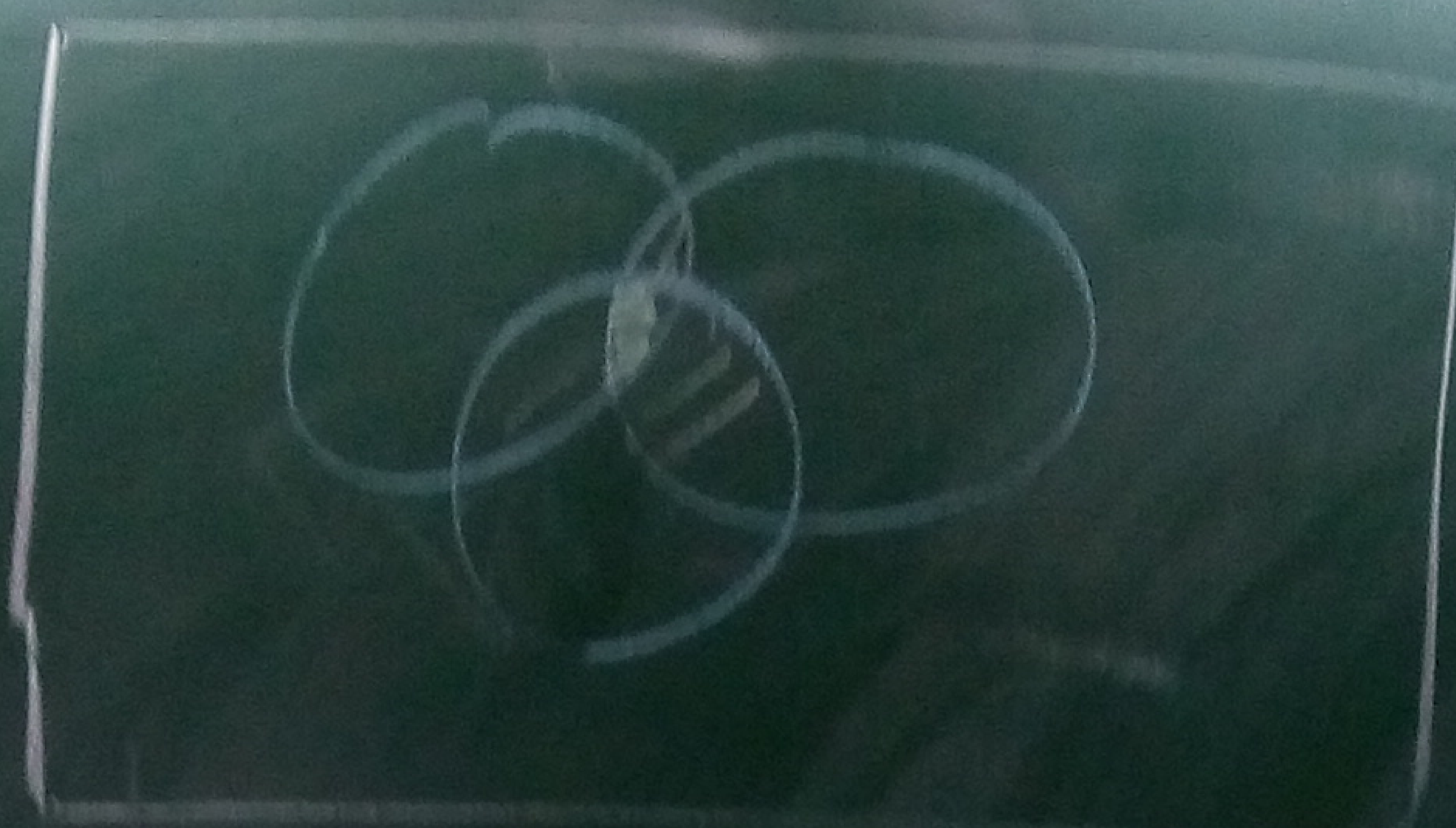
$n=2$



$$N(\phi) = |U| - |A_1| - |A_2| + |A_1 \cap A_2|$$

$S=\phi$ $\quad S=\{1\}$ $\quad S=\{2\}$ $\quad S=\{1,2\}$

$n=3$



$$N(\phi) = |U| - |A_1| - |A_2| - |A_3| + |A_1 \cap A_2| + |A_2 \cap A_3| + |A_3 \cap A_1| - |A_1 \cap A_2 \cap A_3|$$

$n \geq 1$

$x \in N$

$n \geq 1$ (sketch)

$x \in N(\phi)$ The right hand side should sum to 1.

are not in any A_i $\longrightarrow$ corresponds to $S = \phi \Rightarrow 1$.

$T \subseteq [n]$ and $x \in A_i \ \forall \ i \in T$

$S \subseteq T$, $|S| = 2m$ or $|S| = 2m+1$

are in some A_i

$(-1)^{|S|} = 1$ $\qquad$ $(-1)^{|S|} = -1$

$$\sum_{S \subseteq T} (-1)^{|S|} = \sum_{k=0}^{|T|} (-1)^k \, {}^{|T|}C_k = (1-1)^{|T|} = 0$$