$\mathbb{N}$ - What is natural numbers?

Peano's Axioms

$\mathbb{N}$ is a set with the following properties

(1) $\mathbb{N}$ has a distinguished element, 1 (say)

(2) $\exists$ a distinguished map $\sigma : \mathbb{N} \to \mathbb{N}$

(3) σ is one-one

(4) there is no $n \in \mathbb{N}$ such that $\sigma(n) = 1$
(not surjective)

(5)

$\boxed{\text{Principle of Induction (I)}}$

Define $\boxed{+}$

- for all

- for all

(5) $S \subseteq \mathbb{N}$ such that

Principle of Induction (I)

ⓐ $1 \in S$

ⓑ if $n \in S$ then $\sigma(n) \in S$

Then $S = \mathbb{N}$

Check

(i) Associativity $+$

(ii) Commutative $+$

(iii) Cancellative

$x + y = y + z$ then

$x = z$

(iv) Distributive $+, \cdot$

(i) A $\cdot$ ✓

C $\cdot$ ✓

- $n \in \mathbb{N}$ $n \neq 1$ there is $m \in \mathbb{N}$ $\sigma(m) = n$

Define $\boxed{+}$ by recursive relations

- for all $n \in \mathbb{N}$ $n + 1 = \sigma(n)$
- for all $m, n \in \mathbb{N}$ $n + \sigma(m) = \sigma(m+n)$

$\boxed{\cdot}$ by recursive relations

$\forall n \in \mathbb{N}$ $n \cdot 1 = n$

$\forall m, n \in \mathbb{N}$ $n \cdot \sigma(m) = n \cdot m + n$

ay)

Def

We

Order on $\mathbb{N}$

Def: $n, m \in \mathbb{N}$ We say $n < m$ if $\exists\ k \in \mathbb{N}$ $m = n + k$

Read:- $n \le m$ as $n < m$ or $n = m$

. **Well ordered** $m, n \in \mathbb{N}$

One of these three holds

$m < n$ or $m = n$ or $n < m$

(need $m \neq m + k$ $k \in \mathbb{N}$)

II Principle of Induction

$S \subseteq \mathbb{N}$ $S \neq \phi$ $\exists\, m \in S$ (Unique) such that $m \le n$ $\forall\, n \in S$

($S \subseteq \mathbb{N}$ $S \neq \phi$ then S has a least element)

III Principle of Induction

$P(n) \equiv$ mathematical statement for $n \in \mathbb{N}$

Assume: - $P(1)$ is true
- If $P(n)$ is true then $P(n+1)$ is true

Then $P(n)$ is true for all $n \in \mathbb{N}$

IV Principle of Induction

$P(n) \equiv$ mathematical statement for $n \in \mathbb{N}$

Assume - $P(1)$ is true
- if $n > 1$ $P(k)$ is true for $1 \le k < n$ then $P(n)$ is true.

Then $P(n)$ is true for all $n \in \mathbb{N}$

$\mathbb{N} \hookrightarrow \mathbb{Z} \hookrightarrow \mathbb{Q} \hookrightarrow (\mathbb{R}, +, \cdot)$

complete ordered field

Define (2) $\underline{\mathbb{N}}$: intersection of all $S \subseteq \mathbb{R}$

with property (i) $1 \in S$

(ii) $x \in S$ then $x+1 \in S$

$\boxed{S = \mathbb{R}}$ ✓

Prove III . $S = \{ n \in \mathbb{N} \mid P(n) \text{ is true} \}$

Definition of S.

- $1 \in S$
- $k \in S \Rightarrow k+1 \in S$ $\Rightarrow S \supseteq \mathbb{N}$ but $S \subseteq \mathbb{N}$

Def of $\mathbb{N}$ (2)

$S = \mathbb{N}$ □

III

verify

. P(1)

. let P(k) b

Then

(+) & (*), P

$\Rightarrow$ P(n) is t

for all

field

III

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad = P(n)$$

Verify
. P(1)

$$1 = \frac{1(2)}{2} \quad — (+)$$

$$1 = 1$$

let P(k) be true $\quad \sum_{i=1}^{k} i = \frac{k(k+1)}{2} \quad — (*)$

Then $\quad \sum_{i=1}^{k+1} i = \sum_{i=1}^{k} i + (k+1)$

$$\overset{(*)}{=} \frac{k(k+1)}{2} + k+1$$

$$= \frac{(k+1)(k+2)}{2} \quad — (*+)$$

$\boxed{(+) \,\&\, (*), \text{ P of I} \Rightarrow P(n) \text{ is true for all } n \in \mathbb{N}}$

IV

$a_1 = 1 \quad a_2 = 8, \quad a_n = a_{n-1} + 2a_{n-2} \quad \forall n \geq 3$

P(n) $\quad a_n = 3\,2^{n-1} + 2(-1)^n \quad \forall n \in \mathbb{N}$

Verify

P(1) $\quad a_1 = 3(2^0) + 2(-1)^1$
$\quad = 1$

P(k) is true $\quad a_m = 3\,2^{m-1} + 2(-1)^m \quad \forall m \leq k$

P(k+1) $\quad a_{k+1} = a_k + 2a_{k-1}$

$$= 3\,2^{k-1} + 2(-1)^k + 2(3\,2^{k-2} + 2(-1)^{k-1})$$

$$= 3\,2^k + 2(-1)^{k+1}$$

Def. ...

Well...

On...