Recall

- Sets — a collection of distinct objects

  "$x \in A$"

  $A \subseteq B$, $B \subseteq A$ $A = B$

- function $f: A \to B$ $\forall a \in A$ assigns a unique element $b \in B$

  Denote $b := f(a)$

- Let S be a set. An order on S is a relation denoted by $<$ with following two properties:

  (i) $x \in S$, $y \in S$ only one of the statement hold $x < y$ or $y < x$ or $x = y$

(ii) If $x, y, z \in S$ $\quad x < y$ & $y < z$ then $x < z$

• An ordered set is a set S equipped with order relation.

Example 4 :-

$\mathbb{N}$ is an ordered set

$\mathbb{Z}$ is an ordered set

$\mathbb{Q}$ is an ordered set

$\mathbb{R}$ is an ordered set

$\boxed{\leq \text{ means } < \text{ or } =}$

Example $S = \mathbb{N}$

$\{1, 3, 5\} = E \subseteq \mathbb{N}$

Take $\beta = 10$

$x \leq \beta \ \forall x \in E$

(ie $1 \leq 10$, $3 \leq 10$, $5 \leq 10$)

$\therefore$ E is Bounded above

Definition :- (1) Let S be an ordered set & $E \subseteq S$. If there is a $\beta \in S$ such that $x \leq \beta$ for all $x \in E$ then we say E is bounded above.

- $\beta \in S$ is called an upper bound of E.

(2) Let S be an ordered set, $E \subseteq S$, & E is bounded above. Suppose there is an $\alpha \in S$ with the following properties

(i) α is an upper bound of E (ie $x \leq \alpha \ \forall x \in E$)

(ii) If $\gamma < \alpha$ then γ is not an upper bound of E. (ie There is $x \in E$ such that $\gamma < x$)

Then $\alpha \in S$ is called LEAST UPPER BOUND

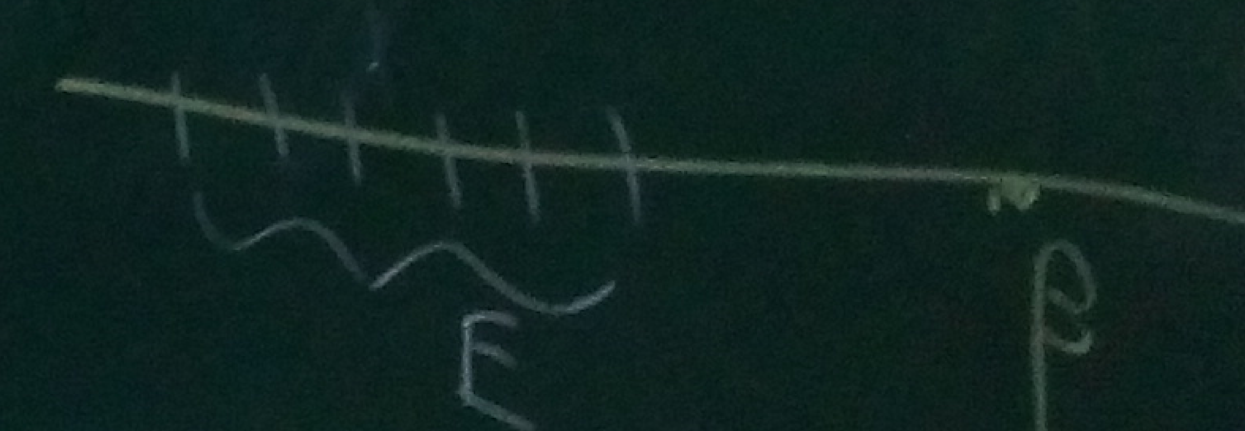
- If $\alpha \in S$ is the least upper bound of E and $\alpha \in E$ then α is the maximum of E

Example 4 (contd)

$S = \mathbb{N}$, $F = \{x \in \mathbb{N} \mid x \text{ is even}\}$ is NOT bounded above.

(*) $E \subseteq \mathbb{N}$ and E is bounded above then $\exists\, \alpha \in E$ such that $x \leq \alpha \ \forall x \in E$.

Proof: (Ex) Idea:



(*) holds for $\mathbb{Z}$.

(**) – $S = \mathbb{Q}$. $E \subseteq \mathbb{Q}$ and E is bounded above then $\exists$ $\alpha \in E$ such that $x \le \alpha$ $\forall x \in E$

$E = \{\frac{1}{2}, \frac{5}{4}, \frac{5}{4}, 2\}$ $\alpha = 2$ ✓

$F = \{p \in \mathbb{Q} \mid p = 1 - \frac{1}{n} \text{ for } n \in \mathbb{N}\}$, $G = \{p \in \mathbb{Q} \mid p < 2\}$ $\alpha = 2$ $\alpha \notin F$,

$\alpha = 1$ is least upper bound of F in $\mathbb{Q}$ $\alpha \notin F$.

F 0 $\frac{1}{2}$ $\frac{3}{4}$ 1

(***) $S = \mathbb{Q}$ $E \subseteq \mathbb{Q}$ and E is bounded above

then $\exists$ $\alpha \in \mathbb{Q}$ such that $x \leq \alpha$ $\forall x \in E$

$H = \{ x \in \mathbb{Q} \mid x^2 < 2 \}$ $\beta = 5$ an upper bound of H

- No least upper bound in $\mathbb{Q}$

(2) Let S be an ordered set, $E \subseteq S$, & E is bounded above. Suppose there is an $\alpha \in S$ with the following properties

(i) α is an upper bound of E (ie $x \leq \alpha$ $\forall x \in E$)

(ii) If $\gamma < \alpha$ then γ is not an upper bound of E.

Then $\alpha \in S$ is called THE LEAST UPPER BOUND (ie There is $x \in E$ such that $\gamma < x$)