To find minima $\quad f : \mathbb{R}^n \to \mathbb{R}$

$$\boxed{\begin{array}{l}\text{What } f\text{?} \\ z^{(k)} \to z^*\end{array}}$$

- $z^{(0)}$ - initial seed

- $z^{(k)} = z^{(k-1)} - t_k \nabla f(z^{(k-1)})$

$t_k > 0 \quad , \quad k = 1, \ldots, T$

$\boxed{\text{Gradient descent}}$

1.

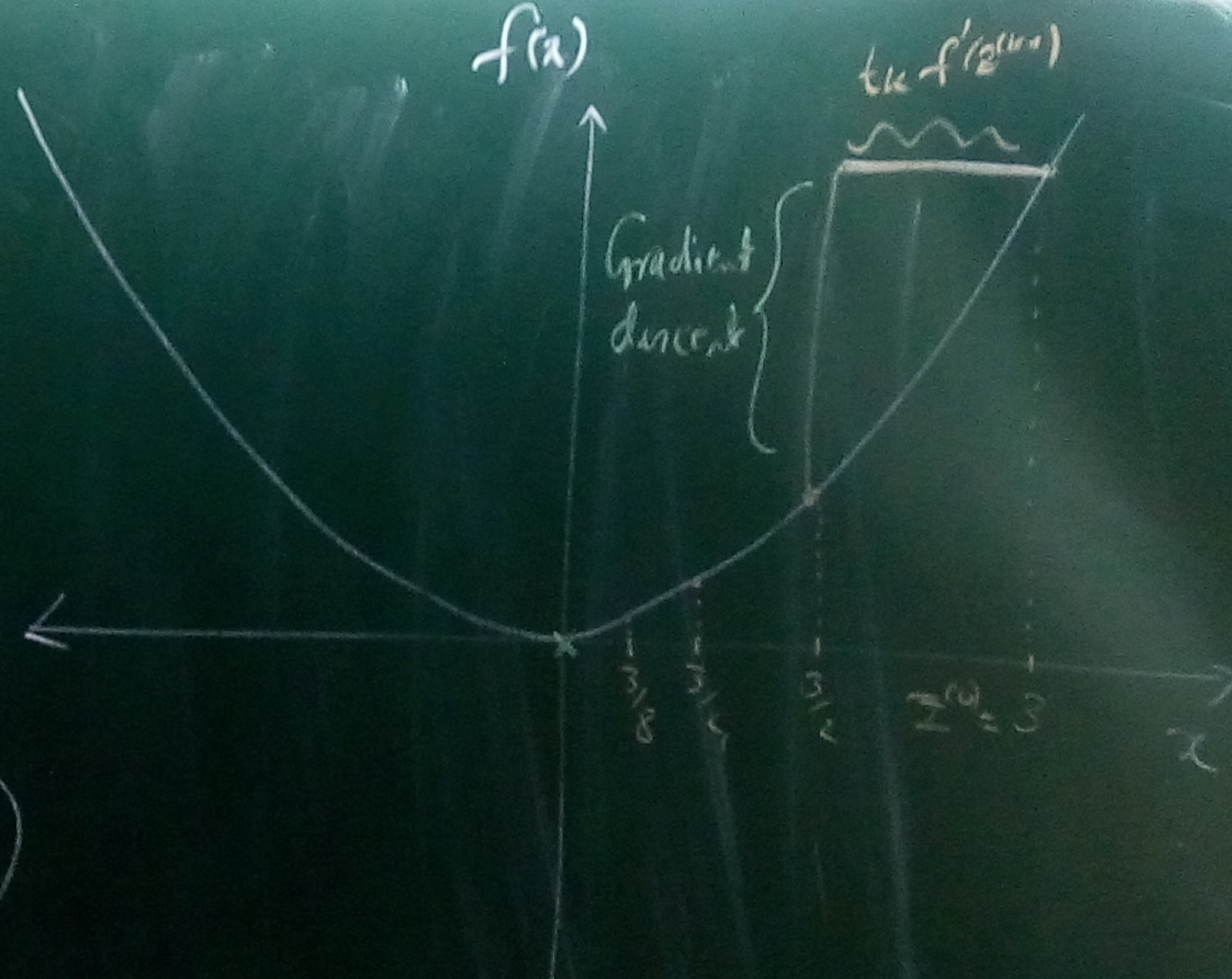
(b)

(c) z

1. $f(x) = x^2$

(a) $f'(x) = 2x$

(b)



(c) $$z^{(k)} = z^{(k-1)} - \frac{1}{4} f'(z^{(k-1)})$$

$$z^{(k)} = z^{(k-1)} - \frac{1}{4} 2 \cdot z^{(k-1)} \Rightarrow z^{(k)} = \frac{1}{2} z^{(k-1)} \quad \forall k \geq 1$$

$k=1, \quad z^{(1)} = \frac{3}{2}$

$k=2, \quad z^{(2)} = \frac{3}{4}$

$k=3, \quad z^{(3)} = \frac{3}{8}$

$k=4 \quad z^{(4)} = \frac{3}{16}$

(e) $z^{(0)} = -4$

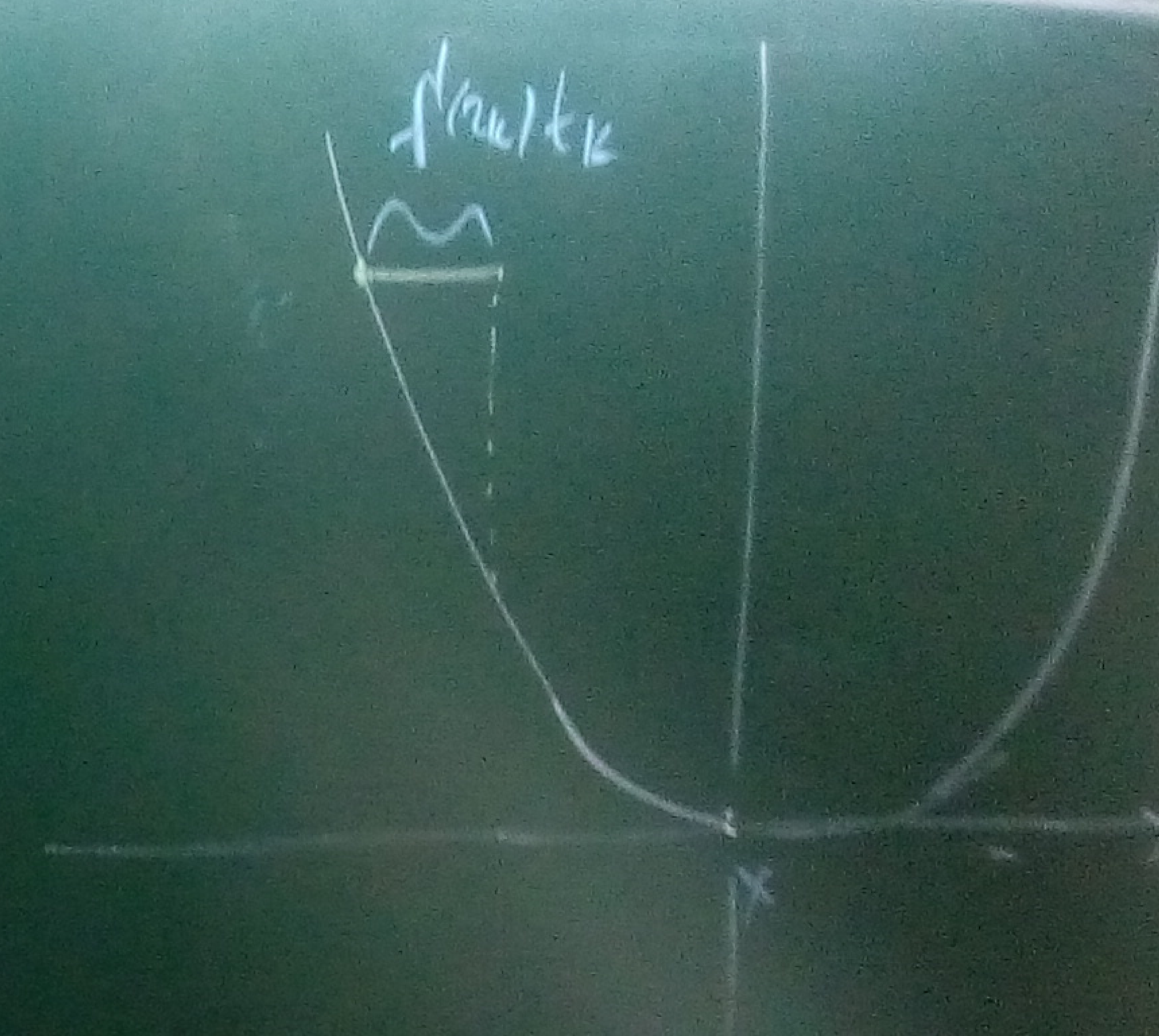
$z^{(k)} = \frac{1}{2} z^{(k-1)}$

$k=1, \quad z^{(1)} = \frac{3}{2} \quad , f(z^{(1)}) = 9/4$

$k=2, \quad z^{(2)} = 3/4 \quad , f(z^{(2)}) = 9/16$

$k=3 \quad , \quad z^{(3)} = 3/8 \quad , f(z^{(3)}) = \frac{9}{64}$

$k=4 \quad z^{(4)} = 3/16 \quad f(z^{(4)}) = \frac{9}{256}$



(e) $z^{(0)} = -4$

$z^{(k)} = \frac{1}{2} z^{(k-1)}$, $z^{(1)} = -2$, $z^{(2)} = -1$, $z^{(3)} = -\frac{1}{2}$, $z^{(4)} = -1/4$

(f) $z^{(k)} = \frac{1}{2} z^{(k-1)}$

, $z^{(k)}$ are decreasing ,

, $z^{(k)} \geq 0$

So $\{z^{(k)}\}_{k \geq 1}$ - converging.

First show by induction

$$z^{(1)} = \frac{1}{2} z^{(0)}$$

$n = k \quad z^{(k)} = \left(\frac{1}{2}\right)^{k} z^{(0)}$

inductive step

$n = k+1 \quad z^{(k+1)} = \frac{1}{2} z^{(k)} = \left(\frac{1}{2}\right)^{k+1} z^{(0)}$

assumption of gradient descent

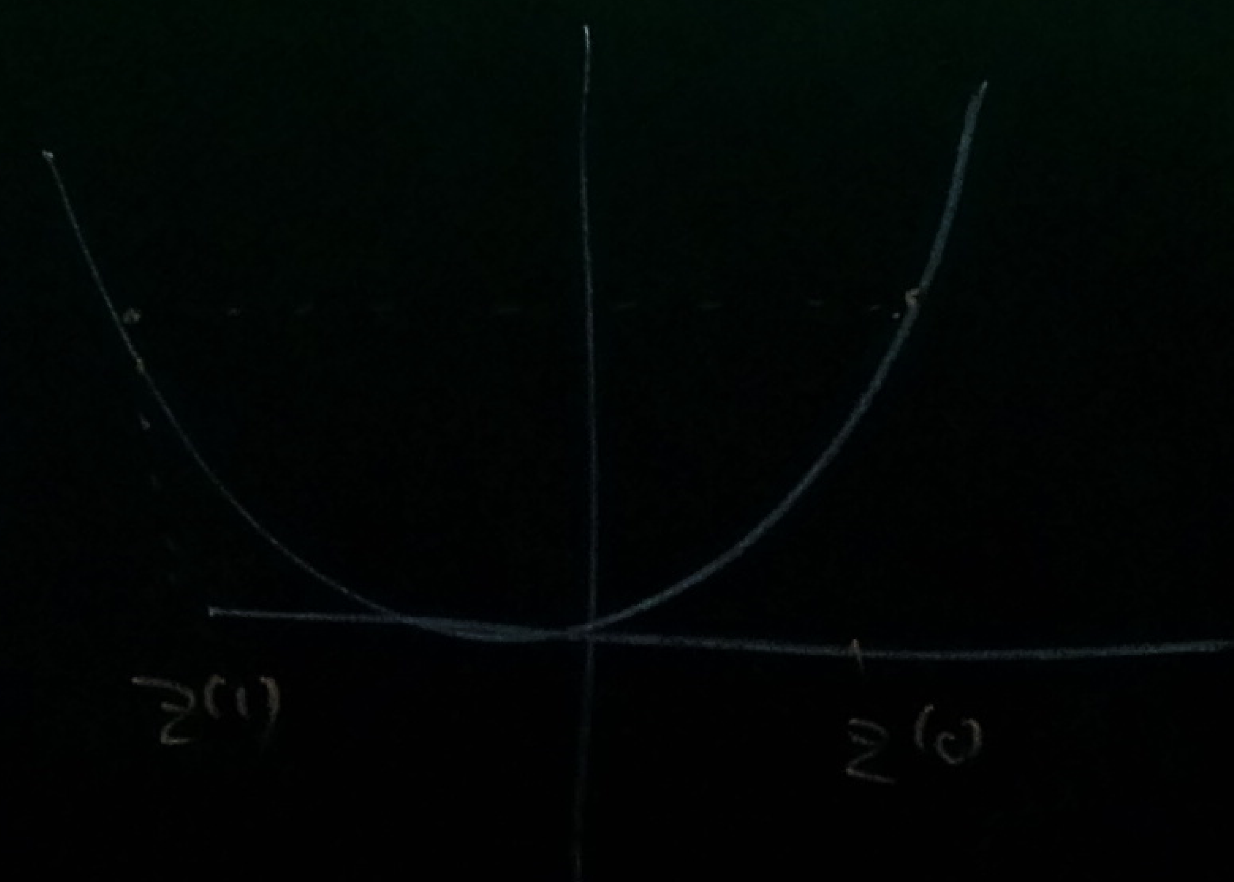
$\Rightarrow z^{(k)} = \frac{1}{2^k} z^{(0)}$

as $z^{(0)} \in \mathbb{R}$

$\left(\frac{1}{2^k} \to 0 \text{ as } k \to \infty, \ \frac{1}{2^k} z^{(0)} \to 0 \text{ as } k \to \infty \right)$

- Choice of t_k :-

$t_k = 1$ $\qquad z^{(k)} = z^{(k-1)} - 1\,2z^{(k-1)} = -z^{(k-1)}$



$t_k = a$ $\quad \underline{0<a<1}$ $\qquad z^{(k)} = z^{(k-1)} - 2a\,z^{(k-1)} = (1-2a)\,z^{(k-1)}$

$|1-2a|<1$

$\iff -1 < 1-2a < 1 \iff 0<a<1$

$\Rightarrow z^{(k)} = (1-2a)^k z^{(0)}$

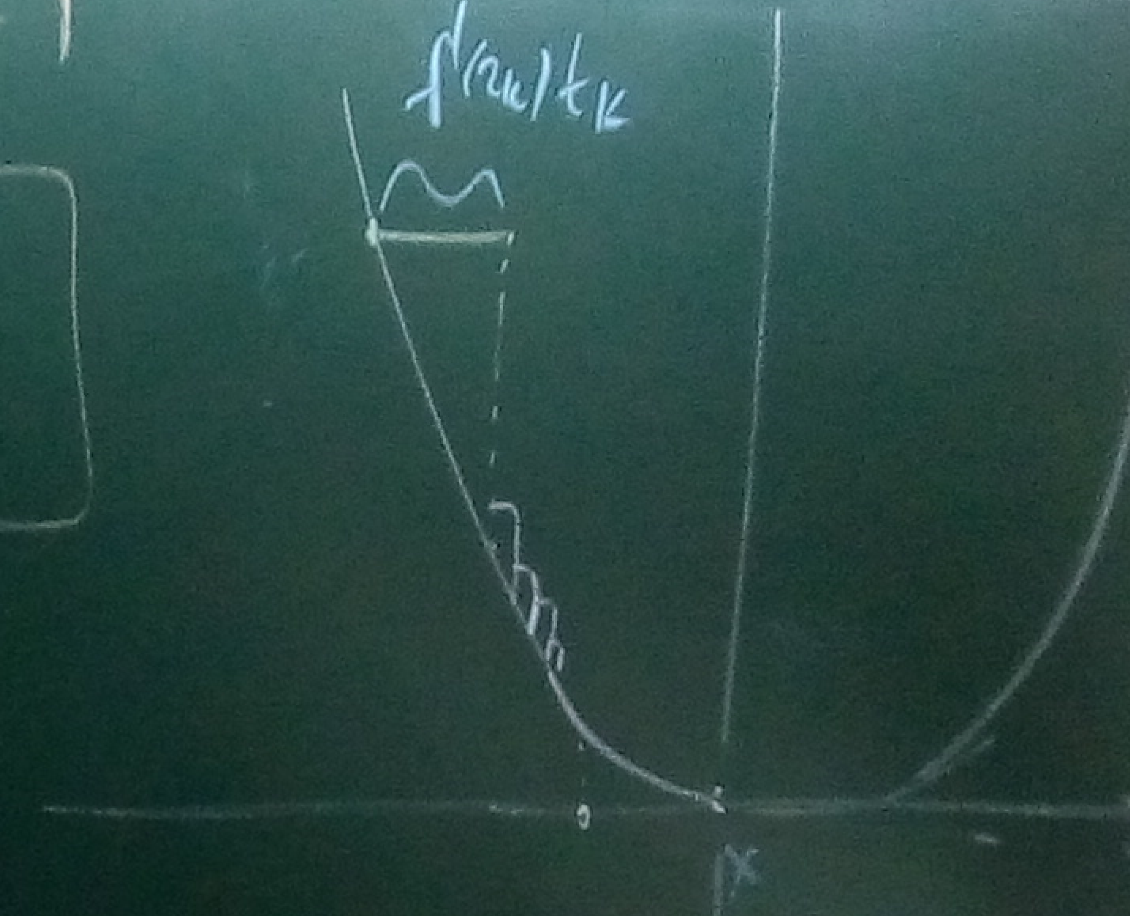
$$t_k = a^k \quad \boxed{\sum_{k=1}^{\infty} t_k = ?}$$

$$0 < a < 1$$

$$\Rightarrow \quad z^{(k)} = z^{(k-1)} - a^k \, 2 z^{(k-1)}$$

$$= (1 - 2a^k) \, z^{(k-1)}$$

$$\boxed{q_k = \prod_{i=1}^{k} b_i}$$



$\Rightarrow$ (induction)

$$z^{(k)} = \prod_{i=1}^{k} (1 - 2a^i) \, z^{(0)}$$

$$\boxed{Q : \ z^{(k)} \to 0 \ ?}$$

(e)

$$z^{(0)} = -4$$

$$z^{(k)} = \frac{1}{2} z^{(k-1)}, \quad z^{(1)} = -2, \quad z^{(2)} = -1, \quad z^{(3)} = -\frac{1}{2}, \quad z^{(4)} = -\frac{1}{4}$$

(f) $z^{(k)} = \frac{1}{2} z^{(k-1)}$

- $z^{(k)}$ are decreasing
- $z^{(k)} \geq 0$

So $\{z^{(k)}\}_{k \geq 1}$ - converges.