

Correction: Ordered field - Properties

$$x < y, z \in \mathbb{R} \Rightarrow x + z < y + z.$$

$$x > 0, y > 0 \Rightarrow xy > 0.$$

Result 4: If x, y are real numbers then

$$2xy < x^2 + y^2 \quad \& \quad xy \leq \left(\frac{x+y}{2}\right)^2$$

If $x \geq 0$ and $y \geq 0$ then $\sqrt{xy} \leq \frac{x+y}{2}$.

(Ex) $\leftarrow$ Equality holds in each only when $x=y$.

Proof:

[Ordered
Propo

$$a < b \& c < b \Rightarrow a < c$$

check. $z \in \mathbb{R}$ then $z^2 \geq 0$

Proof. $0 \leq (x-y)^2 = x^2 + y^2 - 2xy$

[Ordered field]
Property

$$2xy \leq x^2 + y^2$$

$$2xy + 2xy \leq x^2 + y^2 + 2xy$$

$$\Rightarrow 4xy \leq x^2 + y^2 + 2xy = (x+y)^2$$

$$a < b \text{ \& } c > 0 \\ \Rightarrow ac < bc$$

$$\Rightarrow xy \leq \frac{(x+y)^2}{4}$$

$$\Rightarrow xy \leq \left(\frac{x+y}{2}\right)^2$$

[Result 1] $x \geq 0, y \geq 0$
 $xy \geq 0$ and $\left(\frac{x+y}{2}\right)^2 \geq 0$

$$\dots 0 \leq xy \leq \left(\frac{x+y}{2}\right)^2$$

$$\Rightarrow \sqrt{xy} \leq \sqrt{\left(\frac{x+y}{2}\right)^2}$$

$$\Rightarrow \sqrt{xy} \leq \frac{x+y}{2}$$

(Corollary 4) - If $xy > 0$

$$\frac{2xy}{x+y} \leq \sqrt{xy} \leq \frac{x+y}{2}$$

Proof: Result 4 $\Rightarrow \sqrt{xy} \leq \frac{x+y}{2} \text{ --- } (*)$
(order field)

$$x, y > 0 \quad \downarrow \Rightarrow \frac{2\sqrt{xy}}{x+y} > 0$$

$\therefore$ multiplying $(*)$ on both sides by $\frac{2\sqrt{xy}}{x+y}$

$$\Rightarrow \frac{2\sqrt{xy}}{x+y} \cdot \sqrt{xy} \leq \frac{2\sqrt{xy}}{x+y} \cdot \frac{x+y}{2}$$

$$\Rightarrow \frac{2xy}{x+y} \leq \sqrt{xy} \text{ --- } (+)$$

$(*)$ and $(+)$ imply the result. $\square$

Yesterday's puzzle

$$d = v_1 \times 15 \Rightarrow v_1 = d/15$$

$$d = v_2 \times 25 \Rightarrow v_2 = d/25$$

Average speed

$$= \frac{2d}{15+25} = \frac{1}{(1/15 + 1/25)^{1/2}}$$

$$\downarrow \frac{2v_1v_2}{v_1+v_2}$$

Harmonic mean

Sets - I think of a set
as a collection of distinct objects
with a precise description that provides
a way of deciding whether a given
object is in the set or not.

- objects in a set are Elements
of the set

Let A be a set

$x \in A$ (ie x belongs to A , or x is an element of A)

Let B be another set

$A \subseteq B$ if $x \in A \Rightarrow x \in B$.
(whenever $x \in A$ then $x \in B$)

$\mathbb{N}$ - natural numbers = " $\{1, 2, 3, \dots\}$ " = Peano's Axioms

$\mathbb{Z}$ - Integers = " $\{0\} \cup \mathbb{N} \cup \{-k \mid k \in \mathbb{N}\}$ "

$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$

$\mathbb{Q}$ - Rational numbers, $\mathbb{R}$ - Real numbers

Let A, B be two sets s.t. $A=B$ if both the sets have the same elements $\Rightarrow \begin{cases} A \subseteq B \text{ and } B \subseteq A \\ x \in A \text{ if and only if } x \in B \end{cases}$

Proper subset of A is a subset of A that is not A itself
 $\phi = \text{empty set}$ - "has no elements" $\phi \subseteq A$

Power set of A - set of all subsets of A

Specifying a set $\{x \in A \mid \text{specify a condition on } x\}$

A "universe"

E.g. $\{x \in \mathbb{R} \mid 3x^2 + 4x + 5 = 0\}$

Example 1

$$S = \{x \in \mathbb{R} \mid x^2 < x\}$$

$$T = \{x \in \mathbb{R} \mid 0 < x < 1\}$$

Claim: $S = T$

Proof

$$x \in T \Rightarrow 0 < x < 1$$

As $x > 0$

$$\Rightarrow 0 < x^2 < x$$

(ordered field)
Property

$$\Rightarrow x^2 < x$$

$$\Rightarrow x \in S.$$

Let $x \in S$

$$\Rightarrow x^2 < x$$

(*)

$$\Rightarrow x^2 - x < x - x$$

$$\Rightarrow x^2 - x < 0$$

As $x^2 - x = x(x-1)$

$\Rightarrow x(x-1) < 0$

[ordered field $ab < 0 \Rightarrow$ Either $a < 0$ & $b > 0$ OR $a > 0$ & $b < 0$]

$\Rightarrow$ Either $x < 0$ and $x-1 > 0$

OR $x > 0$ and $x-1 < 0$

As $x < 0$ and $x > 1$ is not possible, this implies

$x > 0$ and $x < 1 \Rightarrow 0 < x < 1 \Rightarrow x \in I$

$x \in I$
 $\Rightarrow$
 $x \in I$
 $\Rightarrow$
 $x \in I$

Example 2

$$S = \{x \in \mathbb{R} \mid x^2 + x - 6 = 0\}$$

$$T = \{-3, 2\}$$

Claim: $S = T$

Proof

$$x \in T \Rightarrow x = -3 \text{ or } x = 2$$

Observe that if $x = -3$, $x^2 + x - 6 = (-3)^2 + (-3) - 6 = 0$
or if $x = 2$, $x^2 + x - 6 = (2)^2 + (2) - 6 = 0$

$$\Rightarrow x \in S$$

$$x \in S \Rightarrow \text{imitate proof of day 1} \Rightarrow x = -3 \text{ or } x = 2$$

$$\Rightarrow x \in T$$

Functions

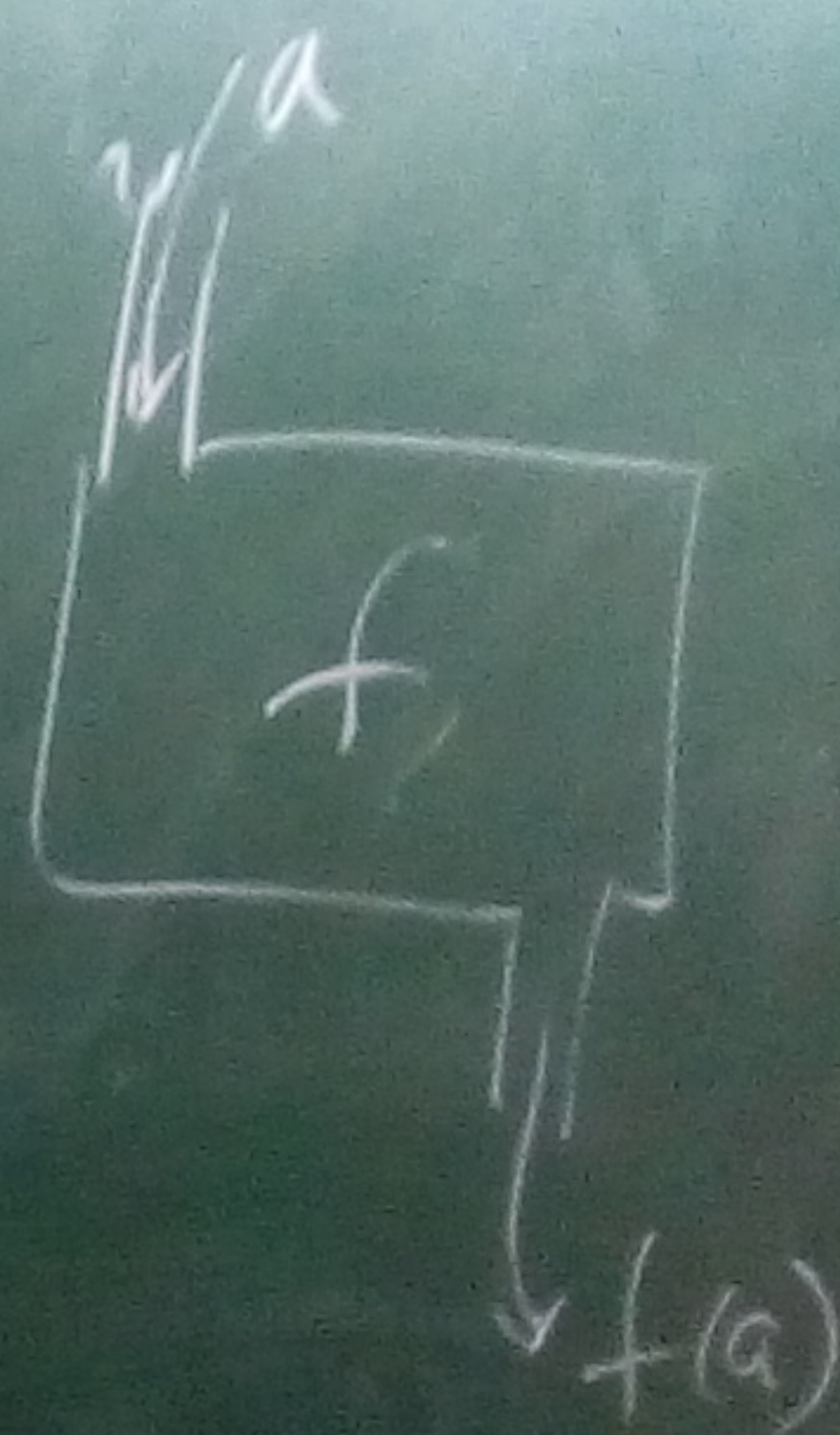
A, B be two sets

" $f: A \rightarrow B$ " $\equiv$ a function f from set

Notation

A to B assigns to

every $a \in A$ a unique $b \in B$ and
 $b = f(a)$



Well-defined :- Rule that assigns
an element in A to an element B should
satisfy definition of a function.

Example 3: $f: \mathbb{R} \rightarrow \mathbb{R}$

$$f(x) = \frac{x}{1+x^2}$$

$$B = \{x \in \mathbb{R} \mid \exists a \in \mathbb{R} \text{ s.t. } f(a) = x\}$$

Find B .

Proof

$$B = [-\frac{1}{2}, \frac{1}{2}]$$

$$|f(x)| \leq \frac{1}{2}$$

Let $a \in \mathbb{R}$ & suppose we show

$$\text{Then } B \subseteq [-\frac{1}{2}, \frac{1}{2}]$$

Now $a \in \mathbb{R}$

$$f(a) = \frac{a}{1+a^2}$$

$$|f(a)| = \frac{|a|}{1+a^2}$$

$$\text{If } |f(a)| \leq \frac{1}{2} \Leftrightarrow \frac{|a|}{1+a^2} \leq \frac{1}{2} \Leftrightarrow 2|a| \leq 1+a^2$$

$$\Leftrightarrow 0 \leq 1+a^2-2|a| = (1-|a|)^2$$

$$\text{As } 1-|a|^2 \geq 0 \Rightarrow |f(a)| \leq \frac{1}{2} \text{ for all } a \in \mathbb{R}$$

$$y \in B$$

$$\Rightarrow y = f(a)$$

$$\text{for some } a \in \mathbb{R}$$

$$\Rightarrow |y| \leq \frac{1}{2}$$

$$\Rightarrow B \subseteq \left[-\frac{1}{2}, \frac{1}{2}\right]$$

$$\text{Let } y \in \left[-\frac{1}{2}, \frac{1}{2}\right] \text{ check if } \exists a \in \mathbb{R} \text{ such that } f(a) = y$$

$$\text{If } \exists a \in \mathbb{R} \text{ st. } \frac{a}{1+a^2} = y$$

Observe $f(0) = 0$. Assume $y \neq 0$ & $a \neq 0$
 $(\because f(a) = 0 \Leftrightarrow a = 0)$

$$\frac{a}{1+a^2} = y \Leftrightarrow a^2 y + y - a = 0$$

$$\Rightarrow ya^2 - a + y = 0$$

$$\Rightarrow a = \frac{1 \pm \sqrt{1-4y}}{2y}$$

$$0 \text{ } 8a \neq 0$$

?

0

$$(*) \quad 1 - 4y^2 \geq 0 \quad \text{is true as} \quad |y| \leq \frac{1}{2} \quad \left(\begin{array}{l} (=) \quad 2|y| \leq 1 \quad (=) \quad 4|y|^2 \leq 1 \\ (=) \quad 1 - 4y^2 \geq 0 \end{array} \right)$$

$$f(a)$$

$$y \in B \Rightarrow y = f(a) \text{ for some } a \in A$$

$$\text{let } y \in$$