Analysis: Cantor Curve

Key words:

1. Cantor Set
2. nowheredense
3. disconnected
4. Continuous functions
5. Onto functions
6. Space filling curves

1. Let $I = [0,1]$. Let $I_1 = I_{11} = (\frac{1}{3}, \frac{2}{3})$ be the open middle third interval of I. Next, let I_{21} and I_{22} be the two open middle third intervals of $I - I_1$. Let $I_2 = I_{21} \cup I_{22}$. For $j \geq 3$ and $k = 1, 2, 3 \ldots, 2^{j-1}$, let I_{jk} be the open middle third intervals of $I - \cup_{k=1}^{j-1} I_k$ and let $I_j = \cup_{k=1}^{2^{j-1}} I_{jk}$. Finally, let $C = I - \cup_{j=1}^{\infty} I_j$. C is called the cantor set.
   (a) Show that C is compact.
   (b) Show that C is an uncountable closed set.
2. Let f be a continuous real function on $\mathbb{R}$ with the following properties: $0 \leq f(t) \leq 1, f(t+2) = f(t)$ for every t, and

$$f(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \frac{1}{3} \\ 3t - 1 & \text{if } \frac{1}{3} < t < \frac{2}{3} \\ 1 & \text{if } \frac{2}{3} \leq t \leq 1, \end{cases},$$

   Let

$$x(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n-1}t) \text{ and } y(t) = \sum_{n=1}^{\infty} 2^{-n} f(3^{2n}t).$$

   Prove that $\Phi(t) = (x(t), y(t))$ is continuous and maps $[0,1]$ onto $[0,1] \times [0,1]$. Can you show that Φ maps the cantor set onto $[0,1] \times [0,1]$.

Mathematician's in Focus Cantor