Daubechies Wavelet/Scaling Filters

Daubechies $\{h_l\}$'s defined via squared gain functions:

$$\mathcal{H}^{(D)}(f) \equiv 2\sin^{\mathrm{L}}(\pi \mathrm{f}) \sum_{l=0}^{\frac{L}{2}-1} \binom{\frac{L}{2}-1+l}{l} \cos^{2l}(\pi f)$$

$$\mathcal{G}^{(D)}(f) = 2\cos^{L}(\pi f) \sum_{l=0}^{\frac{L}{2}-1} \binom{\frac{L}{2}-1+l}{l} \sin^{2l}(\pi f)$$

can obtain $\{h_l\}$ or, equivalently, $\{g_l\}$
via spectral factorization (Daubechies, 1992)

Daubechies Wavelet/Scaling Filters

- Transfer function for different $\{g_l\}$ given by

$$G(f) = \sqrt{\mathcal{G}^{(D)}(f)}e^{i\theta^{(G)}(f)}$$

  - $\mathcal{G}^{(D)}(\cdot)$ fixed for a given L
  - phase function $\theta^{(G)}(\cdot)$ yields particular $\{g_l\}$'s
  - spectral factorization chooses different $\theta^{(G)}(\cdot)$'s

Extremal Phase $\{g_l\}$

- One of two factorizations in Daubechies (1992)

- Note: extremal phase same as minimum phase

- Denote these filters by $\{g_l^{(ep)}\}$.

Extremal Phase $\{g_l^{(ep)}\}$

- If $\{g_l\}$ & $\{g_l^{(ep)}\}$ have same $\mathcal{G}^{(D)}(\cdot)$,

$$\sum_{l=0}^{m} g_l^2 \leq \sum_{l=0}^{m} \left[g_l^{(ep)}\right]^2 \quad \text{for } m = 0, \ldots, L-1$$

- left-hand side defines partial energy sequence
- partial energy builds up fastest for $\{g_l^{(ep)}\}$ ('front loaded')
- Lth order filter called D(L) scaling filter

Zero Phase Filters

- consider filter $\{u_l\} \longleftrightarrow U(\cdot)$ and let $\{u_l^\circ\} \longleftrightarrow U^\circ_{(\cdot)}$ be $\{u_l\}$ periodized to length N,
- let $\{X_t\}$ be time series of length N with DFT $\{\mathcal{X}_k\}$
- let $\{Y_t\}$ be $\{X_t\}$ circularly filtered with $\{u_l^\circ\}$:

$$Y_t \equiv \sum_{l=0}^{N-1} u_l^\circ X_{t-l \bmod N}, \quad t = 0, \ldots, N-1$$

Zero Phase Filters

- write $U(f) = |U(f)|e^{i\theta(f)}$ & suppose $\theta(f) = 0$;
  i.e., $\{u_l\}$ has zero phase
- since $U(f) = |U(f)|$, have $U_k^\circ = |U_k^\circ|$, so

$$Y_t = \frac{1}{N} \sum_{k=0}^{N-1} |U_k^\circ| \mathcal{X}_k e^{i2\pi kt/N}$$

- thus $|U_k^\circ|\mathcal{X}_k$ has same phase as $\mathcal{X}_k$, but amplitude can be different

Zero Phase Filters

- practical implications
  - sinuoidal components of $\{Y_t\}$ align with similar components in $\{X_t\}$
  - ‘events’ in $\{Y_t\}$ are aligned with events in $\{X_t\}$

Zero Phase Filters

Example with zero phase:

$$u_{1,l} = \begin{cases} 1/2, & l = 0; \\ 1/4, & l = +1, -1; \\ 0, & otherwise; \end{cases} \longleftrightarrow \cos^2(\pi f),$$

Example without zero phase:

$$u_{2,l} = \begin{cases} 1/2, & l = 0, 1; \\ 0, & otherwise; \end{cases} \longleftrightarrow e^{-i\pi f} \cos(\pi f),$$

Linear Phase Filters: I

Consider circularly shifting $\{Y_t\}$ by ν units:

$$Y_t^{(\nu)} \equiv Y_{t+\nu \bmod N}, \quad t = 0, \ldots, N-1$$

Example $\nu = 2$ & $N = 11$ yields

$$Y_8^{(2)} = Y_{8+2 \bmod 11} = Y_{10}$$

$\{Y_t^{(\nu)}\}$ advanced version of $\{Y_t\}$ if $\nu > 0$

$\{Y_t^{(\nu)}\}$ delayed version of $\{Y_t\}$ if $\nu < 0$

Zero Phase Filters

Note:

$$\begin{aligned}
\boldsymbol{Y}_t^{(\nu)} = \boldsymbol{Y}_{t+\nu \bmod N} &= \sum_{l=0}^{N-1} u_l^{\circ} \boldsymbol{X}_{t+\nu-l \bmod N} \\
&= \sum_{l=-\nu}^{N-1-\nu} u_{l+\nu}^{\circ} \boldsymbol{X}_{t-l \bmod N} \\
&= \sum_{l=0}^{N-1} u_{l+\nu \bmod N}^{\circ} \boldsymbol{X}_{t-l \bmod N}
\end{aligned}$$

thus can advance filter output by advancing filter

Linear Phase Filters

- $U^{(\nu)}(f) = e^{i2\pi f\nu}U(f)$
- If $\{u_l\}$ has zero phase, i.e. $U(f) = |U(f)|$, implies $\{u_l^{(\nu)}\}$ has $\theta^{(\nu)}(f) = 2\pi f\nu$,
  $\{u_l^{(\nu)}\}$ **said to have linear phase**
- **Conclusion:** can convert linear phase filter to zero phase filter by appropriate advancing (assumes ν is an integer)

Linear Phase Filters

- Example:

$$u_{3,l} = \begin{cases} 1/2, & l = 1; \\ 1/4, & l = 0, l = 2; \\ 0, & otherwise; \end{cases} \longleftrightarrow \cos^2(\pi f) e^{-i2\pi f}$$

  – $\theta_3(f) = -2\pi f$, i.e., linear phase with $\nu = -1$

  – advancing $\{u_{3,l}\}$ by 1 unit yields zero phase filter

Daubechies Least Asymmetric Filter

Definition of LA(L) scaling filter: factorization of $\mathcal{G}^{(D)}(\cdot)$ with $\theta^{(G)}(\cdot)$ such that

$$\max_{-1/2 \leq f \leq 1/2;\ \tilde{\nu}=0,\pm 1,\ldots} \left| \theta^{(G)}(f) - 2\pi f \tilde{\nu} \right|$$

is as small as possible

- let $\{g_l^{(la)}\}$ denote resulting LA(L) scaling filter
- let ν be the $\tilde{\nu}$ that minimizes the above; i.e., $\theta^{(G)}(f) \approx 2\pi f \nu$

Daubechies Least Asymmetric Filter

Let $\{h_l^{(la)}\}$ denote corresponding wavelet filter

$H(f) = e^{-i2\pi f(L-1)+i\pi} G(\frac{1}{2} - f)$, so

$$\begin{aligned}
\theta^{(H)}(f) &= -2\pi f(L-1) + \pi + \theta^{(G)}(\tfrac{1}{2} - f) \\
&\approx -2\pi f(L-1) + \pi + \pi\nu - 2\pi f\nu \\
&= -2\pi f(L-1+\nu) + \pi(\nu+1) \\
&= -2\pi f(L-1+\nu)
\end{aligned}$$

if ν is odd. Thus ν odd $\Longrightarrow \{h_l^{(la)}\}$ approximately linear phase

Daubechies Least Asymmetric Filter

For tabulated LA coefficients, have

$$\nu = \begin{cases} -\frac{L}{2} + 1, & \text{if} L = 8, 12, 16, 20 \text{(i.e., } \frac{L}{2} \text{ is even);} \\ -\frac{L}{2}, & \text{if} L = 10 \text{or} 18; \\ -\frac{L}{2} + 2, & \text{if} L = 14, \end{cases}$$

So ν is indeed odd for all 7 LA scaling filters

Conclusion: LA wavelet filters also $\approx$ linear phase

Appropriate shift to get zero phase is
$-(L - 1 + \nu)$

Shifts for Higher Level Filters

$$\{g_{j,l}\} \longleftrightarrow G_j(f) = \prod_{l=0}^{j-1} G(2^l f)$$

$$\{h_{j,l}\} \longleftrightarrow H_j(f) = H(2^{j-1} f) G_{j-1}(f)$$

phase functions for $\{g_{j,l}\}$ & $\{h_{j,l}\}$ given by

$$\theta_j^{(G)}(f) = \sum_{l=0}^{j-1} \theta^{(G)}(2^l f) \ \&$$

$$\theta_j^{(H)}(f) = \theta^{(H)}(2^{j-1} f) + \sum_{l=0}^{j-2} \theta^{(G)}(2^l f),$$

Shifts for Higher Level Filters

So $\{g_{j,l}\}$ & $\{h_{j,l}\}$ are $\approx$ linear phase also

$$\theta_j^{(G)}(f) \approx 2\pi f \nu_j^{(G)} \quad \text{with} \quad \nu_j^{(G)} \equiv (2^j - 1)\nu$$
$$\theta_j^{(H)}(f) \approx 2\pi f \nu_j^{(H)} \quad \text{with} \quad \nu_j^{(H)} \equiv -(2^{j-1}[L-1] +$$

Can achieve approximate zero phase by advancing filters $|\nu_j^{(G)}|$ or $|\nu_j^{(H)}|$ units

Summary of Daubechies Filters

- Daubechies class of scaling filters $\{g_l\}$ satisfy

$$\mathcal{G}^{(D)}(f) = 2\cos^L(\pi f) \sum_{l=0}^{\frac{L}{2}-1} \binom{\frac{L}{2}-1+l}{l} \sin^{2l}(\pi f)$$

  where $\mathcal{G}^{(D)}(\cdot)$ is the squared gain function for $\{g_l\}$

- for given width L, several filters with same $\mathcal{G}^{(D)}(\cdot)$ **(these differ only in their phase functions)**

Extremal (or minimum) phase Filters

- yields D(L) scaling filters, denoted as $\{g_l^{(ep)}\}$
- maximizes increase of partial energy sequence

Least asymmetric Filters

- yields LA(L) scaling filters, denoted as $\{g_l^{(la)}\}$
- approximately zero phase with shift ν
- $\{h_l^{(la)}\}$'s $\approx$ zero phase (shift is $-(L-1+\nu)$)

Coiflet Filters $C(L)$

- second class of filters yielding DWT describable as generalized differences of weighted averages (due to Daubechies, but suggested by R. Coifman)

- defined for widths $L = 6, 12, 18, 24$ and 30

- involve $L/3$ embedded differencing operations (as opposed to $L/2$ for Daubechies filters)

Coiflet Filters

- can express squared gain function as

$$\mathcal{H}^{(c)}(f) = \mathcal{D}^{\frac{L}{3}}(f)$$

$$\left(\sum_{l=0}^{\frac{L}{6}-1} \binom{\frac{L}{6} - 1 + l}{l} \cos^{2l}(\pi f) + \cos^{\frac{L}{3}}(\pi f) F(f) \right)^2$$

where $F(\cdot)$ chosen so that

$$\mathcal{H}^{(c)}(f) + \mathcal{H}^{(c)}(f + \tfrac{1}{2}) = 2$$

($F(\cdot)$ cannot be expressed in closed form)

Example using ECG Data: I

- $N = 2048$ samples collected at rate of 180 samples/second; i.e., $\Delta t = \frac{1}{180}$ second
- 11.38 seconds of data in all
- set $t_0 = 0.31$ seconds for plotting purposes

Example using ECG Data: II

To quantify how well DWTs summarize $\mathbf{X}$, form normalized partial energy sequence (NPES):

Given $\{U_t : t = 0, \ldots, N-1\}$, square and order such that

$$U_{(0)}^2 \geq U_{(1)}^2 \geq \cdots \geq U_{(N-2)}^2 \geq U_{(N-1)}^2$$

NPES defined as

$$C_n \equiv \frac{\sum_{u=0}^{n} U_{(u)}^2}{\sum_{u=0}^{N-1} U_{(u)}^2}, \quad n = 0, 1, \ldots, N-1$$

Choice of Wavelet Filter

- ANOVA : can use Haar or D(4) or pick L via a simple procedure
- MRA: pick $\{h_l\}$ like 'characteristic features'
  - Haar and D(4) usually a poor match;
    LA filters typically better in practice
  - can use NPESs to quantify match between $\{h_l\}$ and characteristics of $\{X_t\}$
- use LA filters if alignment of $W_{j,t}$ important

Other Practical Considerations

- handling $N \neq 2^J$
  - partial DWT just requires $N = M2^{J_0}$
  - can pad with $\bar{X}$ etc.
  - can truncate down to multiple of 2^{J_0}
    - truncate at beginning of series & do analysis
    - truncate at end of series & do analysis
    - combine two analyses together

Other Practical Considerations

- can use specialized pyramid algorithm (at most one special term at each stage)
- choice of level J_0 of partial DWT
  - application dependent (recall ECG example)
  - default: pick J_0 such that circularity influences $< 50\%$ of $\mathbf{W}_{J_0}$ or $\mathcal{D}_{J_0}$
  - note interplay with $N \neq 2^J$