DWT - Pyramid Algorithm

- can describe pyramid algorithm using either
  - linear filtering operations
  - matrix manipulations
- two approaches complementary, so will do both
- filtering approach begins with notion of wavelet filter

The Wavelet Filter

Let $\{h_l : l = 0, \ldots, L-1\}$ be a real-valued filter 3 properties: (Assume $h_l = 0$ for $l < 0$ & $l \geq L$)

$$\sum_{l=0}^{L-1} h_l = 0 \quad \text{(summation to zero)}$$

$$\sum_{l=0}^{L-1} h_l^2 = 1 \quad \text{(unit energy)}$$

The Wavelet Filter

$$\sum_{l=0}^{L-1} h_l = 0 \quad \text{(summation to zero)}$$

$$\sum_{l=0}^{L-1} h_l^2 = 1 \quad \text{(unit energy)}$$

$$\sum_{l=0}^{L-1} h_l h_{l+2n} = \sum_{l=-\infty}^{\infty} h_l h_{l+2n} = 0 \text{ (orthogonality to even shifts)}$$

The Wavelet Filter

- Transfer & squared gain functions for $\{h_l\}$:

$$H(f) \equiv \sum_{l=0}^{L-1} h_l e^{-i2\pi fl}$$

and

$$\mathcal{H}(f) \equiv |H(f)|^2$$

- Orthonormality property equivalent to

$$\mathcal{H}(f) + \mathcal{H}(f + \tfrac{1}{2}) = 2 \quad \text{for all } f$$

The Rescaled Wavelet Filter

$$\tilde{h}_l \equiv h_l/\sqrt{2}$$

satisfies

$$\sum_{l=0}^{L-1} \tilde{h}_l = 0, \quad \sum_{l=0}^{L-1} \tilde{h}_l^2 = \frac{1}{2}$$

and

$$\sum_{l=-\infty}^{\infty} \tilde{h}_l \tilde{h}_{l+2n} = 0$$

The Rescaled Wavelet Filter

Transfer & squared gain functions for $\{\tilde{h}_l\}$:

$$\widetilde{H}(f) \equiv \sum_{l=0}^{L-1} \tilde{h}_l e^{-i2\pi f l} = \frac{1}{\sqrt{2}} H(f)$$

and

$$\widetilde{\mathcal{H}}(f) \equiv |\widetilde{H}(f)|^2 = \frac{1}{2}\mathcal{H}(f)$$

Frequency domain orthonormality property:

$$\widetilde{\mathcal{H}}(f) + \widetilde{\mathcal{H}}(f + \tfrac{1}{2}) = 1 \text{ for all } f$$

Unit Scale Wavelet Coefficients

Filter $\mathbf{X}$ using $\tilde{h}_l \equiv h_l/\sqrt{2}$:

$$\widetilde{W}_{1,t} \equiv \sum_{l=0}^{L-1} \tilde{h}_l X_{t-l \bmod N}, \quad t = 0, \ldots, N-1$$

For $t = 0, \ldots, N/2 - 1$, define

$$W_{1,t} \equiv 2^{1/2} \widetilde{W}_{1,2t+1} = \sum_{l=0}^{L-1} h_l X_{2t+1-l \bmod N},$$

$\{W_{1,t}\}$ forms first $N/2$ elements of $\mathbf{W} = \mathcal{W}\mathbf{X}$
(first $N/2$ elements of $\mathbf{W}$ form subvector $\mathbf{W}_1$)

Unit scale matrix: $\mathbf{W}_1 = \mathcal{W}_1\mathbf{X}$

$$
\begin{aligned}
W_{1,t} \equiv 2^{1/2}\widetilde{W}_{1,2t+1} &= \sum_{l=0}^{L-1} h_l X_{2t+1-l \bmod N} \\
&= \sum_{l=0}^{N-1} h_l^\circ X_{2t+1-l \bmod N} \\
&= \sum_{l=0}^{N-1} h_{2t+1-l \bmod N}^\circ X_l
\end{aligned}
$$

$\{h_l^\circ\}$ is $\{h_l\}$ periodized to length N

Unit scale matrix: $\mathbf{W}_1 = \mathcal{W}_1 \mathbf{X}$

So we must have

$$\mathcal{W}_{0\bullet}^T = \left[h_1^\circ, h_0^\circ, h_{N-1}^\circ, h_{N-2}^\circ, \ldots, h_2^\circ\right]$$

form last $\frac{N}{2} - 1$ rows of $\mathcal{W}_1$ by circular shifting:

$$\mathcal{W}_{t\bullet}^T = \left[\mathcal{T}^{2t} \mathcal{W}_{0\bullet}\right]^T, \quad t = 1, \ldots, \frac{N}{2} - 1,$$

example:

$$\mathcal{W}_{1\bullet}^T = \left[h_3^\circ, h_2^\circ, h_1^\circ, h_0^\circ, h_{N-1}^\circ, h_{N-2}^\circ, \ldots, h_4^\circ\right].$$

$\mathcal{W}$

The other half of the matrix...$\mathcal{W}$

The Scaling Filter

Scaling filter: $g_l \equiv (-1)^{l+1} h_{L-1-l}$

- reverse $\{h_l\}$ & flip sign of every other coefficient
- e.g.: $h_0 = \frac{1}{\sqrt{2}}$ & $h_1 = -\frac{1}{\sqrt{2}} \Rightarrow g_0 = g_1 = \frac{1}{\sqrt{2}}$
- $\{g_l\}$ is 'quadrature mirror' filter for $\{h_l\}$
- inverse relationship: $h_l = (-1)^l g_{L-1-l}$

The Scaling Filter

$$\sum_{l=0}^{L-1} g_l = \sqrt{2} \text{ (summation to } \sqrt{2}\text{)}$$

$$\sum_{l=0}^{L-1} g_l^2 = 1 \text{ (unit energy)}$$

$$\sum_{l=0}^{L-1} g_l g_{l+2n} = \sum_{l=-\infty}^{\infty} g_l g_{l+2n} = 0 \text{ (orthogonality to even shifts)}$$

Scaling Coefficients: Rescaled scaling filters

Define $\tilde{g}_l \equiv g_l/\sqrt{2}$:

$$\sum_{l=0}^{L-1} \tilde{g}_l = 1, \quad \sum_{l=0}^{L-1} \tilde{g}_l^2 = \frac{1}{2} \; \& \; \sum_{l=-\infty}^{\infty} \tilde{g}_l \tilde{g}_{l+2n} = 0$$

Circularly filter $\{X_t\}$ with $\{\tilde{g}_l\}$ to get

$$\widetilde{V}_{1,t} \equiv \sum_{l=0}^{L-1} \tilde{g}_l X_{t-l \bmod N}, \quad t = 0, \ldots, N-1$$

Scaling Coefficients

Define for $t = 0, \ldots, \frac{N}{2} - 1$:

$$V_{1,t} \equiv 2^{1/2}\widetilde{V}_{1,2t+1} = \sum_{l=0}^{L-1} g_l X_{2t+1-l \bmod N}$$

$$= \sum_{l=0}^{N-1} g_l^{\circ} X_{2t+1-l \bmod N},$$

$\{g_l^{\circ}\}$ is $\{g_l\}$ periodized to N.

$\{V_{1,t}\}$ forms last $N/2$ elements of $\mathbf{W} = \mathcal{W}\mathbf{X}$

$\mathcal{V}_1$

- Let $\mathcal{V}_1$ be the $\frac{N}{2} \times N$ matrix whose first row is

$$\left[g_1^\circ, g_0^\circ, g_{N-1}^\circ, g_{N-2}^\circ, \ldots, g_2^\circ\right] \equiv \mathcal{V}_{0\bullet}^T;$$

other rows are given by

$$[\mathcal{T}^{2t}\mathcal{V}_{0\bullet}]^T, \quad t = 1, \ldots, \frac{N}{2} - 1$$

- have $\mathbf{V}_1 = \mathcal{V}_1 \mathbf{X}$ & $\frac{N}{2}$ rows of $\mathcal{V}_1$ are orthonormal.

Orthonormality of $\mathcal{V}_1$ & $\mathcal{W}_1$

So

$$\mathcal{P}_1 \equiv \begin{bmatrix} \mathcal{W}_1 \\ \mathcal{V}_1 \end{bmatrix}$$

is an $N \times N$ orthonormal matrix since

- $\mathcal{P}_1 \neq \mathcal{W}$ except when $N = 2$
- $\mathcal{V}_1$ spans same subspace as lower half of $\mathcal{W}$

End of 1st Stage of Pyramid Algorithm

- synthesis of $\mathbf{X}$ from $\mathcal{P}_1$:

$$\mathbf{X} = \mathcal{P}_1^T \begin{bmatrix} \mathbf{W}_1 \\ \mathbf{V}_1 \end{bmatrix} = \begin{bmatrix} \mathcal{W}_1^T & \mathcal{V}_1^T \end{bmatrix} \begin{bmatrix} \mathbf{W}_1 \\ \mathbf{V}_1 \end{bmatrix} = \mathcal{W}_1^T \mathbf{W}_1 + \mathcal{V}_1^T$$

  - by definition, $\mathcal{D}_1 = \mathcal{W}_1^T \mathbf{W}_1$
  - since $\mathbf{X} = \mathcal{S}_1 + \mathcal{D}_1$, must have $\mathcal{S}_1 = \mathcal{V}_1^T \mathbf{V}_1$
  - also have $\mathcal{R}_1 = \mathcal{D}_1$

Summary of 1st Stage of Pyramid Algorithm

Transforms $\{X_t : t = 0, \ldots, N-1\}$ into wavelet & scaling coefficients

- $\frac{N}{2}$ wavelet coefficients $\{W_{1,t}, t = 0, \ldots, \frac{N}{2} - 1\}$:

-$\mathbf{W}_1$, an $\frac{N}{2} \times 1$ vector

- changes on scale $\tau_1 = 1$

- first level detail $\mathcal{D}_1$

- $\mathcal{W}_1 = \mathcal{B}_1$, an $\frac{N}{2} \times N$ matrix consisting of first $\frac{N}{2}$ rows of $\mathcal{W}$

- with $\mathcal{B}_1$ containing periodized $\{h_l\}$

Summary of 1st Stage of Pyramid Algorithm

Transforms $\{X_t : t = 0, \ldots, N-1\}$ into wavelet & scaling coefficients

- $\frac{N}{2}$ scaling coefficients $\{V_{1,t}, t = 0, \ldots, \frac{N}{2} - 1\}$:

- $\mathbf{V}_1$, an $\frac{N}{2} \times 1$ vector

- averages on scale $\lambda_1 = 2$
- first level smooth $\mathcal{S}_1$
-$\mathcal{V}_1 = \mathcal{A}_1$, an $\frac{N}{2} \times N$ matrix spanning same subspace as last $\frac{N}{2}$ rows of $\mathcal{W}$
- with $\mathcal{A}_1$ containing periodized $\{g_l\}$

2nd Stage of Pyramid Algorithm: I

- Treat 'scale 2' process $\{V_{1,t} : t = 0, \ldots, \frac{N}{2} - 1\}$ like 'scale 1' process $\{X_t : t = 0, \ldots, N - 1\}$:

$$W_{2,t} \equiv \sum_{l=0}^{L-1} h_l V_{1,2t+1-l \bmod N/2}, \quad t = 0, \ldots, \frac{N}{4} - 1$$

and

$$V_{2,t} \equiv \sum_{l=0}^{L-1} g_l V_{1,2t+1-l \bmod N/2}, \; t=0,\ldots,\frac{N}{4}-1$$

2nd Stage of Pyramid Algorithm: I

Place $W_{2,t}$'s in $\frac{N}{4} \times 1$ vector $\mathbf{W}_2$

- elements $\frac{N}{2}, \ldots, \frac{3N}{4} - 1$ of $\mathbf{W}$
- wavelet coefficients for scale $\tau_2 = 2^{2-1} = 2$

Place $V_{2,t}$'s in $\frac{N}{4} \times 1$ vector $\mathbf{V}_2$

- scaling coefficients for scale $\lambda_2 = 2^2 = 4$

2nd Stage of Pyramid Algorithm

$$\begin{bmatrix} \mathbf{W}_2 \\ \mathbf{V}_2 \end{bmatrix} = \mathcal{P}_2 \mathbf{V}_1 \equiv \begin{bmatrix} \mathcal{B}_2 \\ \mathcal{A}_2 \end{bmatrix} \mathbf{V}_1$$

$\mathcal{B}_2$ & $\mathcal{A}_2$ constructed like $\mathcal{B}_1$ & $\mathcal{A}_1$:

- rows of $\mathcal{B}_2$ have $\{h_l\}$ periodized to length $\frac{N}{2}$ (each row circularly shifted with respect to other rows by multiples of 2)
- rows of $\mathcal{A}_2$ have $\{g_l\}$ periodized to length $\frac{N}{2}$

since $\mathbf{W}_2 = \mathcal{B}_2 \mathbf{V}_1 = \mathcal{B}_2 \mathcal{A}_1 \mathbf{X}$, have $\mathcal{W}_2 = \mathcal{B}_2 \mathcal{A}_1$

2nd Stage of Pyramid Algorithm

Synthesis of $\mathbf{V}_1$ from $\mathbf{W}_2$ & $\mathbf{V}_2$:

$$\mathbf{V}_1 = \mathcal{P}_2^T \begin{bmatrix} \mathbf{W}_2 \\ \mathbf{V}_2 \end{bmatrix} = \begin{bmatrix} \mathcal{B}_2^T & \mathcal{A}_2^T \end{bmatrix} \begin{bmatrix} \mathbf{W}_2 \\ \mathbf{V}_2 \end{bmatrix} = \mathcal{B}_2^T \mathbf{W}_2 + \mathcal{A}_2^T \mathbf{V}_2$$

Since $\mathbf{X} = \mathcal{B}_1^T \mathbf{W}_1 + \mathcal{A}_1^T \mathbf{V}_1$, obtain

$$\begin{aligned} \mathbf{X} &= \mathcal{B}_1^T \mathbf{W}_1 + \mathcal{A}_1^T \mathcal{B}_2^T \mathbf{W}_2 + \mathcal{A}_1^T \mathcal{A}_2^T \mathbf{V}_2 \\ &= \mathcal{W}_1^T \mathbf{W}_1 + \mathcal{W}_2^T \mathbf{W}_2 + \mathcal{A}_1^T \mathcal{A}_2^T \mathbf{V}_2 \end{aligned}$$

2nd Stage of Pyramid Algorithm

- know from 1st stage that $\mathcal{W}_1^T \mathbf{W}_1 = \mathcal{D}_1$
- have $\mathcal{W}_2^T \mathbf{W}_2 = \mathcal{D}_2$ because it involves $\mathbf{W}_2$
- have $\mathcal{A}_1^T \mathcal{A}_2^T \mathbf{V}_2 = \mathcal{S}_2$ because
  $\mathbf{X} - \mathcal{D}_1 - \mathcal{D}_2 = \mathcal{S}_2$
- define $\mathcal{V}_2 = \mathcal{A}_2 \mathcal{A}_1$ so that $\mathcal{V}_2^T \mathbf{V}_2 = \mathcal{S}_2$

2nd Stage of Pyramid Algorithm

orthnormality of $\mathcal{P}_2$ implies

$$\|\mathbf{V}_1\|^2 = \|\mathbf{W}_2\|^2 + \|\mathbf{V}_2\|^2,$$

while from stage 1 we have

$$\|\mathbf{X}\|^2 = \|\mathbf{W}_1\|^2 + \|\mathbf{V}_1\|^2,$$

thus yielding

$$\|\mathbf{X}\|^2 = \|\mathbf{W}_1\|^2 + \|\mathbf{W}_2\|^2 + \|\mathbf{V}_2\|^2$$

Summary of 2nd Stage of Pyramid Algorithm

Transforms $\{V_{1,t} : t = 0, \ldots, \frac{N}{2} - 1\}$ into wavelet & scaling coefficients

- $\frac{N}{4}$ wavelet coefficients $\{W_{2,t}, t = 0, \ldots, \frac{N}{4} - 1\}$:

- $\mathbf{W}_2$, an $\frac{N}{4} \times 1$ vector

- changes on scale $\tau_2 = 2$

- second level detail $\mathcal{D}_2$

- $\mathcal{W}_2 = \mathcal{B}_2\mathcal{A}_1$, an $\frac{N}{4} \times N$ matrix consisting of rows $\frac{N}{2}$ to $\frac{3N}{4} - 1$ of $\mathcal{W}$

Summary of 2nd Stage of Pyramid Algorithm

Transforms $\{V_{1,t} : t = 0, \ldots, \frac{N}{2} - 1\}$ into wavelet & scaling coefficients

- $\frac{N}{4}$ scaling coefficients $\{V_{2,t}, t = 0, \ldots, \frac{N}{4} - 1\}$:

- $\mathbf{V}_2$, an $\frac{N}{4} \times 1$ vector

- averages on scale $\lambda_2 = 4$

- second level smooth $\mathcal{S}_2$

- $\mathcal{V}_2 = \mathcal{A}_2\mathcal{A}_1$, an $\frac{N}{4} \times N$ matrix

spanning same subspace as last $\frac{N}{4}$ rows of $\mathcal{W}$

jth Stage: Pyramid Algorithm

Transforms scale $\lambda_{j-1} = 2^{j-1}$ averages $\{V_{j-1,t} : t = 0, \ldots, \frac{N}{2^{j-1}} - 1\}$ into

- differences on scale $\tau_j = 2^{j-1}$, namely, wavelet coefficients $\{W_{j,t} : t = 0, \ldots, \frac{N}{2^j} - 1\}$

- averages on scale $\lambda_j = 2^j$, namely, scaling coefficients $\{V_{j,t} : t = 0, \ldots, \frac{N}{2^j} - 1\}$

jth Stage: Pyramid Algorithm

In terms of filters (letting $N_j \equiv \frac{N}{2^j}$), for $t = 0, \ldots, N_j - 1$ we have

$$
\begin{aligned}
W_{j,t} &\equiv \sum_{l=0}^{L-1} h_l V_{j-1, 2t+1-l \bmod N_{j-1}}, \\
V_{j,t} &\equiv \sum_{l=0}^{L-1} g_l V_{j-1, 2t+1-l \bmod N_{j-1}},
\end{aligned}
$$

jth Stage of Pyramid Algorithm

- $\mathcal{W}_j = \mathcal{B}_j \mathcal{A}_{j-1} \cdots \mathcal{A}_1$,

  where $\mathcal{W}_j$ & $\mathcal{B}_j$ are $\frac{N}{2^j} \times N$ & $\frac{N}{2^j} \times \frac{N}{2^{j-1}}$

- $\mathbf{W}_j = \mathcal{W}_j \mathbf{X}$ and $\mathbf{W}_j = \mathcal{B}_j \mathbf{V}_{j-1}$

- $\mathcal{D}_j = \mathcal{W}_j^T \mathbf{W}_j$

jth Stage of Pyramid Algorithm

- $\mathcal{V}_j = \mathcal{A}_j \mathcal{A}_{j-1} \cdots \mathcal{A}_1,$

  where $\mathcal{V}_j$ & $\mathcal{A}_j$ are $\frac{N}{2^j} \times N$ & $\frac{N}{2^j} \times \frac{N}{2^{j-1}}$

- $\mathbf{V}_j = \mathcal{V}_j \mathbf{X}$ and $\mathbf{V}_j = \mathcal{A}_j \mathbf{V}_{j-1}$

- $\mathcal{S}_j = \mathcal{V}_j^T \mathbf{V}_j$

jth Stage of Pyramid Algorithm

- analysis of variance at end of stage j:

$$\|\mathbf{X}\|^2 = \sum_{k=1}^{j} \|\mathbf{W}_k\|^2 + \|\mathbf{V}_j\|^2 = \sum_{k=1}^{j} \|\mathcal{D}_k\|^2 + \|\mathcal{S}_j\|^2$$

- multiresolution analysis at end of stage j:

$$\mathbf{X} = \sum_{k=1}^{j} \mathcal{D}_k + \mathcal{S}_j$$

Jth Stage of Pyramid Algorithm

- since $N = 2^J$, algorithm terminates at stage J:

  - $\mathbf{W}_J = [W_{J,0}] = [W_{N-2}]$

  - $\mathbf{V}_J = [V_{J,0}] = [W_{N-1}]$

  - $W_{N-1} = \overline{X}\sqrt{N}$ always,

Discrete Wavelet Transform

$$\mathcal{W} = \begin{bmatrix} \mathcal{W}_1 \\ \mathcal{W}_2 \\ \vdots \\ \mathcal{W}_j \\ \vdots \\ \mathcal{W}_J \\ \mathcal{V}_J \end{bmatrix} = \begin{bmatrix} \mathcal{B}_1 \\ \mathcal{B}_2\mathcal{A}_1 \\ \vdots \\ \mathcal{B}_j\mathcal{A}_{j-1}\cdots\mathcal{A}_1 \\ \vdots \\ \mathcal{B}_J\mathcal{A}_{J-1}\cdots\mathcal{A}_1 \\ \mathcal{A}_J\mathcal{A}_{J-1}\cdots\mathcal{A}_1 \end{bmatrix}$$

Partial DWT

Can choose to stop at $J_0 < J$ repetitions, yielding a partial DWT of level J_0:

$$\begin{bmatrix} \mathcal{W}_1 \\ \mathcal{W}_2 \\ \vdots \\ \mathcal{W}_j \\ \vdots \\ \mathcal{W}_{J_0} \\ \mathcal{V}_{J_0} \end{bmatrix} \mathbf{X} = \begin{bmatrix} \mathcal{B}_1 \\ \mathcal{B}_2\mathcal{A}_1 \\ \vdots \\ \mathcal{B}_j\mathcal{A}_{j-1}\cdots\mathcal{A}_1 \\ \vdots \\ \mathcal{B}_{J_0}\mathcal{A}_{J_0-1}\cdots\mathcal{A}_1 \\ \mathcal{A}_{J_0}\mathcal{A}_{J_0-1}\cdots\mathcal{A}_1 \end{bmatrix} \mathbf{X} = \begin{bmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \\ \vdots \\ \mathbf{W}_j \\ \vdots \\ \mathbf{W}_{J_0} \\ \mathbf{V}_{J_0} \end{bmatrix}$$

$\mathcal{W}_{J_0}$ & $\mathcal{V}_{J_0}$ are $\frac{N}{2^{J_0}} \times N$

Partial DWT

- yields $\frac{N}{2^{J_0}}$ scaling coefficients for scale $\lambda_{J_0} = 2^{J_0}$;
  'complete' DWT yields 1 scaling coefficient
- only requires N to be integer multiple of 2^{J_0};
  complete DWT requires N to be power of 2
- partial DWT more commonly used: scaling coefficients capture 'large scale' components
- setting J_0 is application dependent

Partial DWT

- analysis of variance for partial DWT:

$$\begin{aligned}\hat{\sigma}_X^2 &= \frac{1}{N}\sum_{j=1}^{J_0}\|\mathbf{W}_j\|^2 + \frac{1}{N}\|\mathbf{V}_{J_0}\|^2 - \overline{X}^2 \\ &= \frac{1}{N}\sum_{j=1}^{J_0}\|\mathcal{D}_j\|^2 + \frac{1}{N}\|\mathcal{S}_{J_0}\|^2 - \overline{X}^2\end{aligned}$$

last 2 terms are the sample variance of $\mathcal{S}_{J_0}$

Partial DWT

- multiresolution analysis for partial DWT:

$$\mathbf{X} = \sum_{j=1}^{J_0} \mathcal{D}_j + \mathcal{S}_{J_0}$$

$\mathcal{S}_{J_0}$ represents averages on scale $\lambda_{J_0} = 2^{J_0}$ (includes $\overline{X}$)

Unit scale matrix: $\mathbf{W}_1$

$$W_{1,t} \equiv 2^{1/2}\widetilde{W}_{1,2t+1} = \sum_{l=0}^{L-1} h_l X_{2t+1-l \bmod N}$$

$$= \sum_{l=0}^{N-1} h_l^\circ X_{2t+1-l \bmod N}$$

$\{h_l^\circ\}$ is $\{h_l\}$ periodized to length N

Scaling Coefficients : V_1

Define first level scaling coefficients:

$$V_{1,t} \equiv 2^{1/2}\widetilde{V}_{1,2t+1} = \sum_{l=0}^{L-1} g_l X_{2t+1-l \bmod N}$$

$$= \sum_{l=0}^{N-1} g_l^{\circ} X_{2t+1-l \bmod N},$$

$\{g_l^{\circ}\}$ is $\{g_l\}$ periodized to N

Effect of $\{h_l\}$, $\{g_l\}$

- $\{h_l\}$ is high-pass filter
- $\{g_l\}$ is low-pass because $\{h_l\}$ is high-pass
- same true for all Daubechies wavelet filters

What Kind of Process is $\{V_{1,t}\}$?

$\{X_t\} \longleftrightarrow \{\mathcal{X}_k\}$,

$$X_t = \frac{1}{N}\sum_{k=0}^{N-1} \mathcal{X}_k e^{i2\pi tk/N} = \frac{1}{N}\sum_{k=-\frac{N}{2}+1}^{\frac{N}{2}} \mathcal{X}_k e^{i2\pi tk/N}$$

because $\{\mathcal{X}_k\}$ & $\{e^{i2\pi tk/N}\}$ periodic with period N

What Kind of Process is $\{V_{1,t}\}$?

As $\tilde{g}_l \approx$ low pass with passband $[-\frac{1}{4}, \frac{1}{4}]$, have

$$\widetilde{V}_{1,t} = \sum_{l=0}^{L-1} \tilde{g}_l X_{t-l} \approx \frac{1}{N} \sum_{k=-\frac{N}{4}+1}^{\frac{N}{4}} \mathcal{X}_k e^{i2\pi tk/N}$$

$$V_{1,t} = \sqrt{2}\widetilde{V}_{1,2t+1} \approx \frac{\sqrt{2}}{N} \sum_{k=-\frac{N}{4}+1}^{\frac{N}{4}} \mathcal{X}_k e^{i2\pi(2t+1)k/N},$$

$$= \frac{2}{N} \sum_{k=-\frac{N}{4}+1}^{\frac{N}{4}} \frac{\mathcal{X}_k e^{i2\pi k/N}}{\sqrt{2}} e^{i2\pi tk/(N/2)}$$

$$\equiv \frac{1}{N'} \sum_{k=-\frac{N'}{2}+1}^{\frac{N'}{2}} \mathcal{X}'_k e^{i2\pi tk/N'},$$

$\frac{N}{2} = N'$ and $0 \leq t \leq N' - 1$

What Kind of Process is $\{V_{1,t}\}$?

$$V_{1,t} \approx \frac{1}{N'} \sum_{k=-\frac{N'}{2}+1}^{\frac{N'}{2}} \mathcal{X}'_k e^{i2\pi tk/N'}$$

- $\mathcal{X}'_k$ associated with $f'_k \equiv \frac{k}{N'}$, $-\frac{1}{2} < f'_k \leq \frac{1}{2}$
- $f'_k = \frac{2k}{N} = 2f_k$ so $f_k \in [-\frac{1}{4}, \frac{1}{4}] \Rightarrow f'_k \in [-\frac{1}{2}, \frac{1}{2}]$
- thus $\{\widetilde{V}_{1,t}\}$ is low-pass but $\{V_{1,t}\}$ is 'full-band'

What Kind of Process is $\{W_{1,t}\}$?

As $\tilde{h}_l \approx$ high-pass filter, have

$$\begin{aligned}\widetilde{W}_{1,t} &= \sum_{l=0}^{L-1} \tilde{h}_l X_{t-l} \\ &\approx \frac{1}{N}\left(\sum_{k=-\frac{N}{2}+1}^{-\frac{N}{4}} + \sum_{k=\frac{N}{4}+1}^{\frac{N}{2}}\right) \mathcal{X}_k e^{i2\pi tk/N}\end{aligned}$$

What Kind of Process is $\{W_{1,t}\}$?

$$
\begin{aligned}
W_{1,t} &= 2^{1/2}\widetilde{W}_{1,2t+1} \\
&\approx \frac{\sqrt{2}}{N}\left(\sum_{k=-\frac{N}{2}+1}^{-\frac{N}{4}} + \sum_{k=\frac{N}{4}+1}^{\frac{N}{2}}\right)\mathcal{X}_k e^{i2\pi(2t+1)k/N} \\
&= \frac{2}{N}\left(\sum_{k=-\frac{N}{2}+1}^{-\frac{N}{4}} + \sum_{k=\frac{N}{4}+1}^{\frac{N}{2}}\right)\frac{\mathcal{X}_k e^{i2\pi k/N}}{\sqrt{2}} e^{i2\pi tk/(N/2)}
\end{aligned}
$$

If $\mathcal{X}'_k \equiv \frac{\mathcal{X}_{k+\frac{N}{2}} e^{i2\pi(k+\frac{N}{2})/N}}{\sqrt{2}} = -\frac{\mathcal{X}_{k+\frac{N}{2}} e^{i2\pi k/N}}{\sqrt{2}}$

What Kind of Process is $\{W_{1,t}\}$?

$$W_{1,t} \approx \frac{1}{N'} \sum_{k=-\frac{N'}{2}+1}^{\frac{N'}{2}} \mathcal{X}'_k e^{i2\pi tk/N'}$$

- $\mathcal{X}'_k$ associated with $f'_k \equiv \frac{k}{N'}$, $-\frac{1}{2} < f'_k \leq \frac{1}{2}$
- $f'_k = \frac{2k}{N} = 2(\frac{1}{2} - f_k)$
  so $f_k \in [\frac{1}{4}, \frac{1}{2}] \Rightarrow f'_k \in [0, \frac{1}{2}]$ in *reverse* order
- $\{\widetilde{W}_{1,t}\}$ is high-pass, but $\{W_{1,t}\}$ is 'full-band'

2nd Stage of Pyramid Algorithm: I

- Treat 'scale 2' process $\{V_{1,t} : t = 0, \ldots, \frac{N}{2} - 1\}$ like 'scale 1' process $\{X_t : t = 0, \ldots, N - 1\}$:

$$W_{2,t} \equiv \sum_{l=0}^{L-1} h_l V_{1,2t+1-l \bmod N/2}, \quad t = 0, \ldots, \frac{N}{4} - 1$$

and

$$V_{2,t} \equiv \sum_{l=0}^{L-1} g_l V_{1,2t+1-l \bmod N/2}, \quad t=0,\ldots,\tfrac{N}{4}-1$$

Cascade of Filters

Flow diagram for cascade with 3 filters:

$$\{b_t\} \longrightarrow \boxed{A_1(\cdot)} \xrightarrow{1.} \boxed{A_2(\cdot)} \xrightarrow{2.} \boxed{A_3(\cdot)} \xrightarrow{3.} \{c_t\}$$

if $\{b_t\} \longleftrightarrow B(\cdot)$ & $\{c_t\} \longleftrightarrow C(\cdot)$, then

- output from $A_1(\cdot)$ has DFT $A_1(f)B(f)$
- output from $A_2(\cdot)$ has DFT $A_2(f)A_1(f)B(f)$

So $$\{b_t\} \longrightarrow \boxed{A(\cdot)} \longrightarrow \{c_t\}$$

with $A(f) = A_3(f)A_2(f)A_1(f)B(f)$

Equivalent Filter for 2nd Stage

Path from $\mathbf{X}$ to $\mathbf{W}_2$ almost a filter cascade

- Q: can we obtain $\mathbf{W}_2$ from $\mathbf{X}$ using a single filter?
- A: yes, as the following argument shows

Equivalent Filter for 2nd Stage

- define $\{h_l^\uparrow\} = \{h_0, 0, h_1, 0, \ldots, h_{L-2}, 0, h_{L-1}\}$; note that this filter has width of $2L-1$
- define $\tilde{h}_l^\uparrow = h_l^\uparrow/\sqrt{2}$ & define

$$\widetilde{W}_{2,t} \equiv \sum_{l=0}^{2L-2} \tilde{h}_l^\uparrow \widetilde{V}_{1,t-l \bmod N}, \qquad t = 0, \ldots, N-1;$$

$$
\begin{aligned}
\widetilde{W}_{2,4t+3} &= \sum_{l=0}^{2L-2} \tilde{h}_l^{\uparrow} \widetilde{V}_{1,4t+3-l \bmod N} \\
&= \sum_{l=0}^{L-1} \tilde{h}_l \widetilde{V}_{1,4t+3-2l \bmod N} \\
&= \sum_{l=0}^{L-1} \tilde{h}_l \widetilde{V}_{1,2(2t+1-l)+1 \bmod N} \\
&= \tfrac{1}{2} \sum_{l=0}^{L-1} h_l V_{1,2t+1-l \bmod N/2} = \frac{W_{2,t}}{2};
\end{aligned}
$$

Equivalent Filter for 2nd Stage

Flow diagram, indicating downsampling:

$$\mathbf{X} \longrightarrow \boxed{\{g_l\}} \longrightarrow \boxed{\{h_l^{\uparrow}\}} \longrightarrow 2\widetilde{\mathbf{W}}_2 \underset{\downarrow 4}{\longrightarrow} \mathbf{W}_2$$

Transfer function $H_2(\cdot)$ for equivalent filter is

- transfer function for $\{g_l\}$, i.e., $G(\cdot)$, times
- transfer function for $\{h_l^{\uparrow}\}$, say, $H^{\uparrow}(\cdot)$
- $H^{\uparrow}(f) = H(2f)$, so $H_2(f) = H(2f)G(f)$

Equivalent Filter for 2nd Stage

Denote impulse response sequence for $H_2(\cdot)$ as

$$\{h_{2,l} : l = 0, \ldots, L_2 - 1 = 3L - 3\}$$

Flow diagram with equivalent filter:

$$\mathbf{X} \longrightarrow \boxed{\{h_{2,l}\}} \underset{\downarrow 4}{\longrightarrow} \mathbf{W}_2$$

Equivalent Filter for 2nd Stage

Can write

$$
\begin{aligned}
W_{2,t} &= \sum_{l=0}^{L_2-1} h_{2,l} X_{4(t+1)-1-l \bmod N}, \quad t = 0, \ldots, \tfrac{N}{4} - 1 \\
&= \sum_{l=0}^{N-1} h^{\circ}_{2,l} X_{4(t+1)-1-l \bmod N}, \quad t = 0, \ldots, \tfrac{N}{4} - 1
\end{aligned}
$$

where $\{h^{\circ}_{2,l}\}$ is $\{h_{2,l}\}$ periodized to length N

Equivalent Filters for 2nd Stage: V

Another flow diagram with equivalent filter:

$$\mathbf{X} \longrightarrow \boxed{\{H_2(\tfrac{k}{N})\}} \underset{\downarrow 4}{\longrightarrow} \mathbf{W}_2$$

similarly:

$$\mathbf{X} \longrightarrow \boxed{\{g_l\}} \longrightarrow \boxed{\{g_l^{\uparrow}\}} \underset{\downarrow 4}{\longrightarrow} \mathbf{V}_2$$

$$\mathbf{X} \longrightarrow \boxed{\{G_2(\tfrac{k}{N})\}} \underset{\downarrow 4}{\longrightarrow} \mathbf{V}_2$$

jth Stage: Pyramid

In terms of filters (letting $N_j \equiv \frac{N}{2^j}$), have

$$
\begin{aligned}
W_{j,t} &\equiv \sum_{l=0}^{L-1} h_l V_{j-1,2t+1-l \bmod N_{j-1}}, \qquad t = 0, \ldots, N_j - 1 \\
&= \sum_{l=0}^{L_j-1} h_{j,l} X_{2^j(t+1)-1-l \bmod N},
\end{aligned}
$$

where $\{h_{j,l}\}$ is the jth level equivalent wavelet filter and $L_j = (2^j - 1)(L - 1) + 1$

Equivalent Filter for jth Stage

$\{h_{j,l}\}$ formed by convolution of j filters:

filter 1: $g_0, g_1, \ldots, g_{L-2}, g_{L-1};$

filter 2: $g_0, 0, g_1, 0, \ldots, g_{L-2}, 0, g_{L-1};$

filter 3: $g_0, 0, 0, 0, g_1, 0, 0, 0, \ldots, g_{L-2}, 0, 0, 0, g_{L-1};$

$\vdots$

filter $j-1$: $g_0, 0, \ldots, 0, g_1, 0, \ldots, 0, \ldots, g_{L-2}, 0, \ldots, 0, g_{L-1};$

filter j: $h_0, 0, \ldots, 0, h_1, 0, \ldots, 0, \ldots, h_{L-2}, 0, \ldots, 0, h_{L-1}$

Equivalent Filter for jth Stage

Properties of $\{h_{j,l}\}$:

$$\{h_{j,l} : l = 0, \ldots, L_j - 1\}$$

$$\longleftrightarrow H(2^{j-1}f) \prod_{l=0}^{j-2} G(2^l f) \equiv H_j(f)$$

Nominal passband given by $\frac{1}{2^{j+1}} \leq |f| \leq \frac{1}{2^j}$

Periodized filter for forming rows of $\mathcal{W}_j$:

$$\{h^\circ_{j,l} : l = 0, \ldots, N-1\} \longleftrightarrow \{H(2^{j-1}\tfrac{k}{N}) \prod_{l=0}^{j-2} G(2^l \tfrac{k}{N}) : k = 0, \ldots, N-1\}$$

Equivalent Filter for jth Stage

For the scaling coefficients, have

$$
\begin{aligned}
V_{j,t} &\equiv \sum_{l=0}^{L-1} g_l V_{j-1,2t+1-l \bmod N_{j-1}}, \qquad t = 0, \ldots, N_j - 1 \\
&= \sum_{l=0}^{L_j-1} g_{j,l} X_{2^j(t+1)-1-l \bmod N},
\end{aligned}
$$

where $\{g_{j,l}\}$ is the jth level equivalent scaling filter

Equivalent Filters for jth Stages $\{g_{j,l}\}$

formed by convolution of j filters

- filters 1 to $j-1$ same as used to form $\{h_{j,l}\}$
- filter j:

  $g_0, 0, \ldots, 0, g_1, 0, \ldots, 0, g_{L-2}, 0, \ldots, 0, g_{L-1}$

$$\{g_{j,l} : l = 0, \ldots, L_j - 1\} \longleftrightarrow \prod_{l=0}^{j-1} G(2^l f) \equiv G_j(f)$$

- Nominal passband given by $0 \leq |f| \leq \frac{1}{2^{j+1}}$

Equivalent Filter for jth Stage

- Periodized filter for forming rows of $\mathcal{V}_j$:

$$\{g^{\circ}_{j,l} : l = 0, \ldots, N-1\}$$

$$\longleftrightarrow$$

$$\{\prod_{l=0}^{j-1} G(2^l \tfrac{k}{N}) : k = 0, \ldots, N-1\}$$