Wavelets: An introduction

Let I be a bounded interval on $\mathbb{R}$. The term “wavelet” is used loosely to denote

a function $f : I \longrightarrow \mathbb{R}$

- Oscillatory behaviour.
- Is zero outside or decreases rapidly outside I.
- Dilation or translations give rise to new ones.

Orthonormal Wavelet: Definition

A function $\psi : \mathbb{R} \longrightarrow \mathbb{R}$ such that

1. $\psi(t)$ tends to zero faster than any power of t as $t \longrightarrow \infty$
2. ψ possesses continuous derivatives up to order N, for some positive integer N.
3. For all integers j and n, let

   $$\psi_{jn}(t) = 2^{\frac{j}{2}}\psi(2^j t - n).$$

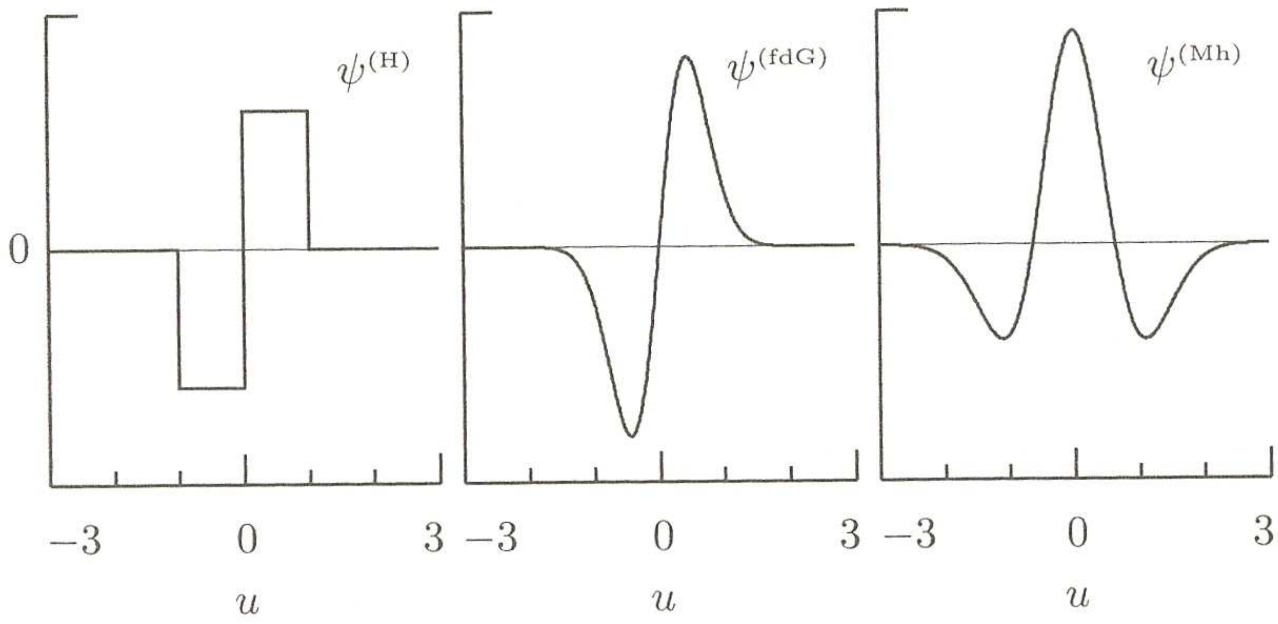
   Then "suitable" functions can be expanded uniquely in a series of ψ_{jn}:

   $$f(t) = \sum_{j,n=-\infty}^{\infty} c_{jn}\psi_{jn}(t).$$

4. The coefficients c_{jn} above are given by

   $$c_{jn} = \int_{-\infty}^{\infty} f(t)\overline{\psi_{jn}(t)}dt.$$

Orthonormal Wavelet : Existence



If P is a polynomial with degree less than or equal to N then

$$\int_{-\infty}^{\infty} \psi_{jn}(t)P(t)dt = 0$$

Orthonormal Wavelet : Existence 86-87

1. Haar wavelet

$$\psi_H(t) = \begin{cases} 1 & 0 < t < \frac{1}{2} \\ -1 & \frac{1}{2} < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

   Too Crude, also (2) is not satisfied.

2. Yves Meyer – took ideas from signal processing and gave an expression in terms of its Fourier Transform Has derivatives of all orders but slow decay at infinity.

3. Guy Battle and Pierre-Gilles Lemarié – constructed a family of orthonormal wavelets one for each order of smoothness N with exponential decay at infinity. They were essentially splines

4. I. Daubechies – constructed a family of orthonormal wavelets with all possible finite orders of smoothness, vanishing outside an interval I.
   Algorithmic computation; limit of recursive equation

Daubechies Wavelet - implications

Assume ψ is a Daubechies Wavelet that satisfies properties (1))-(4).

- ψ_{jn} Vanishes outside I_{jn} obtained by translating I by n and dilating by 2^{-j}.
- Formally Speaking:
  - Frequency of Oscillation (ψ_{jn}) "=" 2^j Frequency of Oscillation (ψ)
  - $j << 0$ represents "low-frequency" oscillation,
  - $j >> 0$ represents "high-frequency" oscillation.

Now,

$$f = \sum_{j,n=-\infty}^{\infty} c_{jn}\psi_{jn}, \;\; c_{jn} = \int_{-\infty}^{\infty} f(t)\overline{\psi_{jn}(t)}dt = \int_{I_{jn}} f(t)\overline{\psi_{jn}(t)}dt$$

Daubechies Wavelet - implications

Rewrite

$$f = \sum_{j=-\infty}^{\infty} S_j, \quad \text{with} \quad S_j = \sum_{n=-\infty}^{\infty} c_{jn}\psi_{jn}$$

- Each S_j focuses on oscillations of frequency roughly 2^j on intervals of length roughly $2-j$
  - c_{jn} depends on values of f in I_{jn}
  - $j >> 0$, each S_j adds another level of detail at the length scale 2^{-j}.
    Looking into a microscope
  - $j << 0$, each S_j represents a view of f on a larger scale
    Looking through the wrong end of a telescope

Daubechies Wavelet - implications

- Smoothness of f near a is reflected in decay of c_{jn} as $j \longrightarrow \infty$ with $a \in I_{jn}$
- Captures the discontinuities in the initial example

$$g_1 : \mathbb{R} \to \mathbb{R}, g_1(t) = \sum_{n=1}^{\infty} \frac{\sin(2\pi nt)}{n}, \quad g_2(t) = \sum_{n=1}^{\infty} \frac{(-1)^n \sin(2\pi nt)}{n}$$

Wavelet - Applications

- Signal processing and two dimensional images
- Time Series Analysis
- Human brain uses something akin to wavelet analysis in processing information it receives from eyes
- Not good for encoding musical signals
- As a subject, wavelets are
  - relatively new (1983 to present)
  - a synthesis of old/new ideas
  - keyword in 22000+ articles & books since 1989 (3100+ since 2003: an inundation of material!!!)