1. Let X and Y be two continuous random variables having the same distribution. Let $f : \mathbb{R} \to \mathbb{R}$ be a piecewise continuous function. Then show that $f(X)$ and $f(Y)$ have the same distribution.

2. Verify 1. for discrete random variables.

3. Let P be the empirical distribution defined by sample observations $X_1, X_2, \ldots, X_n$. In other words, P is the discrete distribution with probability mass function given by the below definition:

   Definition: Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables. The "empirical distribution" based on these is the discrete distribution with probability mass function given by

   $$f(t) = \frac{1}{n}\#\{X_i = t\}.$$

   Let Y be a random variable with distribution P.

   (a) Show that $E(Y) = \bar{X}$.

   (b) Show that $Var(Y) = \frac{n-1}{n}S^2$.

4. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with finite expectation μ, finite variance σ^2, and finite $\gamma = E(X_1 - \mu)^4$. Compute $Var(S^2)$ in terms of μ, σ^2, and γ and show that $Var(S^2) \to 0$ as $n \to \infty$.

5. Let $X_1, X_2, \ldots, X_n$ be i.i.d. random variables with finite expectation μ and finite variance σ^2. let $S = \sqrt{S^2}$, the non-negative root of the sample variance. The quantity S is called the "sample standard deviation". Although $E[S^2] = \sigma^2$, it is not true that $E[S] = \sigma$. In other words, S is not an unbiased estimator for σ. Follow the steps below to see why.

   (a) Let Z be a random variable with finite mean and finite variance. Prove that $E[Z^2] \geq E[Z]^2$ and give an example to show that equality may not hold. (Hint: Consider how these quantities relate to the variance of Z).

   (b) Use (a) to explain why $E[S] \leq \sigma$ and give an example to show that equality may not hold.