

Recall:-

$$0 \leq \alpha \leq 1$$

$$\beta(\theta) = P_{\theta}(X \in R)$$

Rejection region

$$H_0: \theta \in \Theta_0$$

$$H_1: \theta \in \Theta_0^c$$

Size

$$\alpha = \sup_{\theta \in \Theta_0} P_{\theta}(X \in R)$$

level

$$\alpha \geq \sup_{\theta \in \Theta_0} P_{\theta}(X \in R)$$

Unbiased

if

$$\beta(\theta) \geq \beta'(\eta)$$

where  $\theta \in \Theta_0^c$   $\eta \in \Theta_0$

$\beta'$  - power of any other test function

UMP test for class G

-  $\beta(\cdot)$  power function is

uniformly most powerful among class G test if

$$\beta(\theta) \geq \beta'(\theta) \quad \forall \theta \in \Theta_0^c$$

$\beta'$  - power of any test in G



- Simple hypothesis (Hypothesis has only one distribution)

Neyman Pearson lemma provides a characterization of UMP tests.

Theorem 8.3.12 (Neyman-Pearson lemma) Consider testing  $H_0: \theta = \theta_0$  versus  $H_1: \theta = \theta_1$ , using the test with

rejection region  $R$  that satisfies

$$(1) - \begin{cases} x \in R & \text{if } f(x|\theta_1) > k f(x|\theta_0) \\ x \in R^c & \text{if } f(x|\theta_1) < k f(x|\theta_0) \end{cases}$$

where  $k > 0$  &  $f(x|\theta_1)$  is pmf or pdf of data.

$$(2) - \alpha = P_{\theta_0}(x \in R)$$



Then

(a) Any test that satisfies (1) & (2) is a UMP level  $\alpha$ -test

(b) If there is a test satisfying (1) & (2), then  
(i) Every UMP level  $\alpha$  test is also size  $\alpha$ .

(ii) Every UMP level  $\alpha$ -test also satisfies (1)

except <sup>Perhaps</sup> on a set of measure 0, i.e.

$$P_{Q_0}(X \in A) = P_{Q_1}(X \in A) = 0$$

Proof

Let  
Define

Suppose  $\phi$

$\Rightarrow$

Let  $\phi_1$  be

$\beta$ -be

$\beta_1$  be



Proof Let  $f$  be the probability density function.

Define test function  $\phi: S \rightarrow \mathbb{R}$   $\phi(x) = \begin{cases} 1 & x \in R \\ 0 & x \in R^c \end{cases}$

Test -  $\phi$

Suppose  $\phi$  satisfies (1) & (2)

(2)  $\Rightarrow \alpha = P_{\theta_0}(X \in R) \Rightarrow$  As  $(H)_0 = \{\theta_0\} \Rightarrow$  level  $\alpha$  test

Let  $\phi_1$  be another test

$\beta$  - be the power function corresponding to  $\phi$

$\beta_1$  - be the power function corresponding to  $\phi_1$



To show:

$$\beta(\theta_1) \geq \beta(\theta_0)$$

Proof:

$$\textcircled{+} \phi(x) = \begin{cases} 1 & \text{if } [f(x|\theta_1) - k f(x|\theta_0)] > 0 \\ 0 & \text{if } [f(x|\theta_1) - k f(x|\theta_0)] < 0 \end{cases}$$

$$\textcircled{*} (\phi(x) - \phi_1(x))(f(x|\theta_1) - k f(x|\theta_0)) \geq 0$$

$$0 \leq \phi_1(x) \leq 1, \textcircled{+}$$

Case 1  $\phi(x) = 1, 0 \leq \phi_1(x) \leq 1, f(x|\theta_1) - k f(x|\theta_0) > 0$

Case 2  $\phi(x) = 0, 0 \leq \phi_1(x) \leq 1, f(x|\theta_1) - k f(x|\theta_0) < 0$

Case 3  $f(x|\theta_1) = k f(x|\theta_0)$

Inteq

$$\int \phi(x) \dots$$

observe

$$\textcircled{*} \Rightarrow$$



Integrate  $(*)$  across inequality

$$\int (\phi(x) - \phi_1(x)) (f(x|\theta_1) - k f(x|\theta_0)) dx \geq \int 0 dx$$

$$\begin{aligned} & \int \phi(x) f(x|\theta_1) dx - k \int \phi(x) f(x|\theta_0) dx \\ & - \int \phi_1(x) f(x|\theta_1) dx + k \int \phi_1(x) f(x|\theta_0) dx \geq 0 \end{aligned}$$

observe:  $\beta(\theta) = P_\theta(x \in R) = E_\theta(\phi(x)) = \int \phi(x) f(x|\theta) dx$

$$(*) \Rightarrow \begin{aligned} & \beta(\theta_1) - k \beta(\theta_0) \\ & - \beta_1(\theta_1) + k \beta_1(\theta_0) \geq 0 \end{aligned} \quad (**)$$



$\beta(\theta_0) = \alpha$  ( $\because \phi$  satisfies ① & ②)  
Given:-  $\beta_1(\theta_0) \leq \alpha$  ( $\because \phi_1$  is level  $\alpha$ -test),  $\sup_{\theta \in \Theta} P_{\theta_0}(X \in R) \leq \alpha$   
 $\beta_1(\theta) = P_{\theta_0}(X \in R)$

$(\times \times) \quad \beta(\theta_1) - \beta_1(\theta_1) + k(\beta_1(\theta_0) - \beta(\theta_0)) \geq 0$

$\Rightarrow \beta(\theta_1) - \beta_1(\theta_1) \geq -k(\beta_1(\theta_0) - \beta(\theta_0))$   
 $= -k(\beta(\theta_0) - \beta_1(\theta_0))$

$= k(\alpha - \beta_1(\theta_0)) \geq 0$   
 $\Rightarrow \beta(\theta_1) \geq \beta_1(\theta_1) \Rightarrow \phi$  satisfying ① & ② is UMP level  $\alpha$ -test.

① Let  $\phi$  be

(i) Let  $\phi_1$  -

Given:-  $\beta_1(\theta_0)$

$\beta(\theta_0)$

$\Rightarrow \phi$  is also

As  $\phi_1$  is UMP

From (a)  $\phi$  is



If we want size  $\alpha$  then need  $c$  such that

$$\alpha = P_{\theta_0} \left( \sum_{i=1}^n x_i > c \right) = 1 - F_{n, \theta_0}(c)$$

$F_{n, \theta_0}$  - distribution function of  $\text{Poisson}(n\theta_0)$

Choose  $c$  or return  $c$  choose  $k$  such that  $F_{n, \theta_0}(c) = 1 - \alpha$



2(b)  $x_1, \dots, x_n$  iid  $\exp(\theta)$

$$f(x|\theta) = \begin{cases} \theta e^{-\theta x} & x \geq 0 \\ 0 & \text{a.w.} \end{cases}$$

$$H_0: \theta = \theta_0$$

$$H_1: \theta = \theta_1$$

$$\theta_1 > \theta_0$$

$$\underline{k > 0}$$

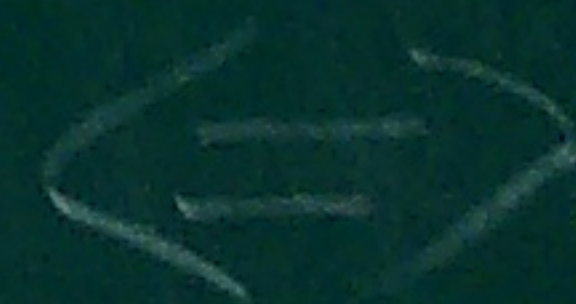
$$\frac{f(x|\theta_1)}{f(x|\theta_0)} = \frac{\prod_{i=1}^n \theta_1 e^{-\theta_1 x_i}}{\prod_{i=1}^n \theta_0 e^{-\theta_0 x_i}} = \frac{\theta_1^n e^{-\theta_1 \sum_{i=1}^n x_i}}{\theta_0^n e^{-\theta_0 \sum_{i=1}^n x_i}}$$

$x \in R$  if  $f(x|\theta_1) > k f(x|\theta_0)$

①  $x \in R^c$  if  $f(x|\theta_1) < k f(x|\theta_0)$   
for some  $k > 0$

$$\alpha = P_{\theta_0}(x \in R)$$

Reject





$x \in R$  if  $f(x) \geq k$   
 $\theta \in R$  if  $f(\theta) \geq k$   
 for some  $k \geq 0$

$$\alpha = P_{\theta_0}(X \in R)$$

$$= \frac{\theta_1^n e^{-\theta_1 \sum_{i=1}^n x_i}}{\theta_0^n e^{-\theta_0 \sum_{i=1}^n x_i}}$$

reject  $H_0$  if  $\frac{\theta_1^n e^{-\theta_1 \sum_{i=1}^n x_i}}{\theta_0^n e^{-\theta_0 \sum_{i=1}^n x_i}} > k$

$$\Leftrightarrow \left(\frac{\theta_1}{\theta_0}\right)^n e^{-\left(\sum_{i=1}^n x_i\right)(\theta_1 - \theta_0)} > k$$

$$\Leftrightarrow \frac{\sum_{i=1}^n x_i \leq \frac{n \log(\frac{\theta_1}{\theta_0}) - \log k}{\theta_1 - \theta_0}}$$



$$\alpha = P_{\theta_0}(X \in R)$$

Find  $k$ : 
$$C = \frac{n \log\left(\frac{\theta_1}{\theta_0}\right) - \log k}{\theta_1 - \theta_0}$$

Such that

$$P_{\theta_0}\left(\sum_{i=1}^n X_i \leq C\right) = \alpha$$

$$\sum_{i=1}^n X_i \sim \text{Gamma}(n, \theta)$$

$$F_{n, \theta_0}(C) = \alpha \quad \text{where}$$

$$F_{n, \theta} \sim \text{Gamma}(n, \theta)$$

$$x \in R \text{ if } f(x|\theta) > k f(x|\theta_0)$$

$$\textcircled{1} x \in R^c \text{ if } f(x|\theta_1) < k f(x|\theta_0) \text{ for some } k > 0$$

$$\textcircled{2} \alpha = P_{\theta_0}(X \in R)$$

$\Leftrightarrow$

$\Leftrightarrow$



Reject  $H_0$  if

$$\frac{\theta_1^n e^{-\theta_1 \sum_{i=1}^n x_i}}{\theta_0^n e^{-\theta_0 \sum_{i=1}^n x_i}} > k$$

$$\Leftrightarrow \left(\frac{\theta_1}{\theta_0}\right)^n e^{-\left(\sum_{i=1}^n x_i\right)(\theta_1 - \theta_0)} > k$$

$$\Leftrightarrow \sum_{i=1}^n x_i \leq \frac{n \log\left(\frac{\theta_1}{\theta_0}\right) - \log k}{\theta_1 - \theta_0}$$



(2) HW<sup>103</sup>  $X \sim \text{Binomial}(100, p)$

$$H_0: p = \frac{1}{2}$$

$$H_1: p \neq \frac{1}{2}$$

Reject  $H_0$  if  $|X - 50| > 10$

(a) What is  $\alpha$ ?

(b) Graph power function



$$\alpha = \sup_{\theta \in \Omega_0} P_{\theta_0}(X \in R) = P_{1/2}(|X - 50| > 10)$$

$$\left( \frac{X - np}{\sqrt{np(1-p)}} \sim Z \right) = P_{1/2} \left( \frac{|X - (100)(1/2)|}{\sqrt{100 \cdot \frac{1}{2}(1 - \frac{1}{2})}} > \frac{10}{\sqrt{100 \cdot \frac{1}{2}(1 - \frac{1}{2})}} \right)$$

$$Z \sim N(0,1) \approx P \left( Z > \frac{10}{\sqrt{100 \cdot \frac{1}{2}(1 - \frac{1}{2})}} \right)$$

$$= 0.045$$

(2) HW'3

$$H_0: p = 1/2$$

$$H_1: p \neq 1/2$$

(a) What is

(b) Graph power



$$\textcircled{b} \quad \beta(p) = \mathbb{P}_p(X \in R)$$

$$= \mathbb{P}_p(|X - 50| > 10)$$

$$= \mathbb{P}_p(X > 60 \text{ or } X < 40)$$

$$= \mathbb{P}_p(X > 60) + \mathbb{P}_p(X < 40)$$

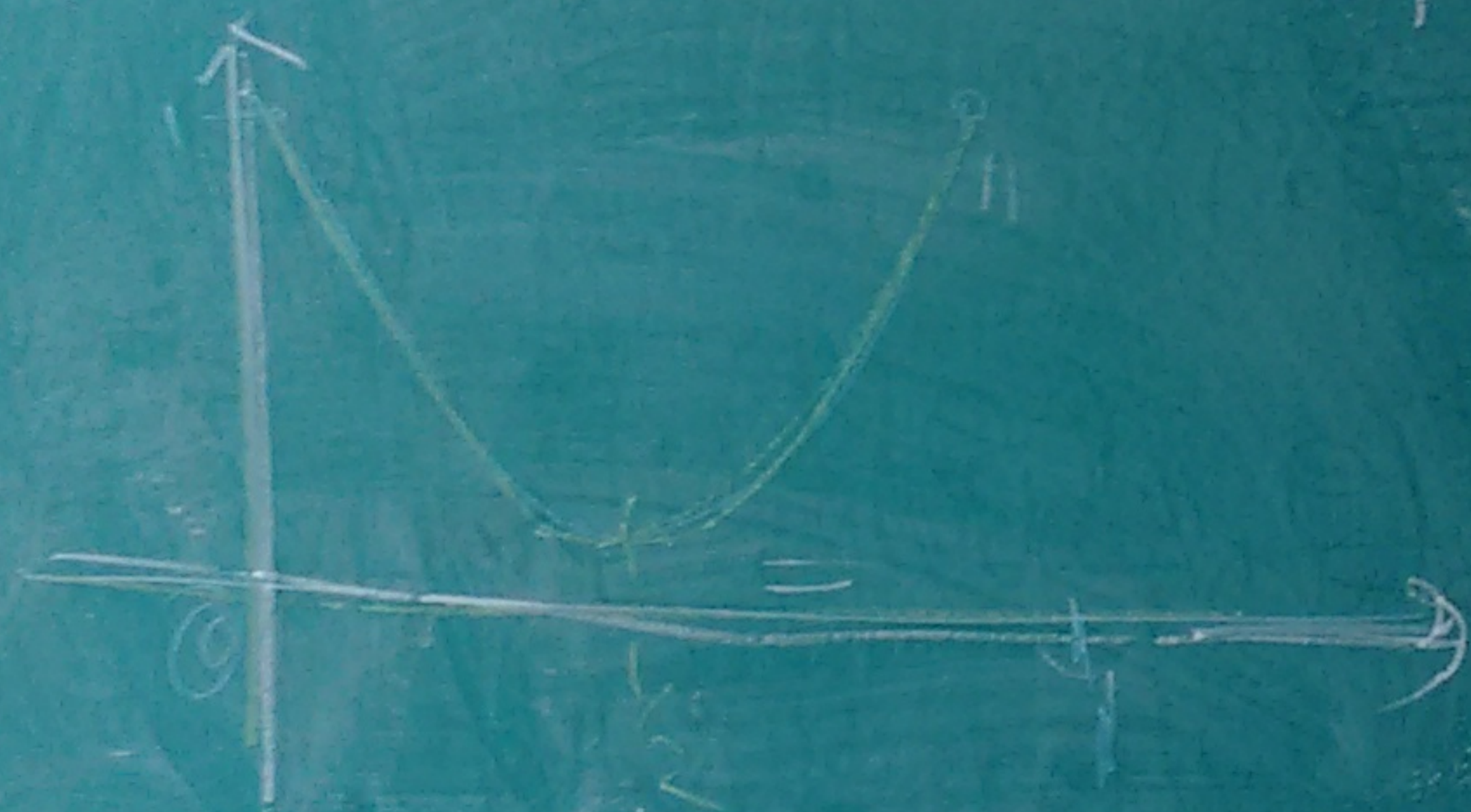
$$= \mathbb{P}_p\left(\frac{X - np}{\sqrt{np(1-p)}} > \frac{60 - np}{\sqrt{np(1-p)}}\right) + \mathbb{P}_p\left(\frac{X - np}{\sqrt{np(1-p)}} < \frac{40 - np}{\sqrt{np(1-p)}}\right)$$

$$\frac{X - np}{\sqrt{np(1-p)}} \sim Z$$



$$\beta(p) \stackrel{n=100}{=} P\left(Z > \frac{60 - 100p}{\sqrt{100 p(1-p)}}\right) + P\left(Z < \frac{40 - 100p}{\sqrt{100 p(1-p)}}\right)$$

$$= P\left(Z > \frac{6 - 10p}{\sqrt{p(1-p)}}\right) + P\left(Z < \frac{4 - 10p}{\sqrt{p(1-p)}}\right)$$



(b)

$$\frac{X - np}{\sqrt{np(1-p)}} \sim Z$$