Recall:

$H_0 : \theta \in \Theta_0$

$H_1 : \theta \in \Theta_0^c$

R - rejection region

$\{X \in R\}$ - reject H_0

$\alpha = \begin{cases} 0.05 \\ 0.10 \\ 0.01 \end{cases}$

Typically

- power function : $\beta(\theta) = P_\theta(X \in R)$

- Size $[=\alpha]$ - $\sup_{\theta \in \Theta_0} P_\theta(X \in R) = \alpha$

- Level $[\alpha]$ - $\sup_{\theta \in \Theta_0} P_\theta(X \in R) \leq \alpha$

- Type I Error : $\theta \in \Theta_0$ $P_\theta(X \in R)$ - Prob of type I error

- Type II Error $\theta \in \Theta_0^c$ $1 - P_\theta(X \in R)$ - Prob of type II error

If we say a test is level α then we are only specifying a control on Probability of type I error. We do not specify anything on Type II error. So it becomes important to specify null & alternative hypothesis appropriately.

Example 8.2.2 (Revisited) $X_1, \ldots X_n$ i.i.d sample from Normal$(\theta, 1)$

$H_0: \theta = \theta_0$ $H_1: \theta \neq \theta_0$ Reject H_0 if $|\bar{X} - \theta_0| > \frac{z_{\alpha/2}}{\sqrt{n}}$

Power function $\beta(\theta) = \mathbb{P}_\theta\left(|\bar{X} - \theta_0| > \frac{z_{\alpha/2}}{\sqrt{n}} \right)$

$$\sup_{\theta \in \Theta_0} \beta(\theta) = \beta(\theta_0) = \mathbb{P}_{\theta_0}\left(|\bar{X} - \theta_0| > \frac{z_{\alpha/2}}{\sqrt{n}} \right)$$

$$(\text{Where } Z \sim N(0,1)) = \mathbb{P}\left(|Z| > z_{\alpha/2} \right)$$

$$= \alpha$$

In fact this test is of size α

Definition 8.3.9 :- A test is said to be unbiased if its power function $\beta(\ )$ satisfies:

$$\left.\begin{array}{ll} \beta(\theta) \geq \beta(\psi) & \text{for } \theta \in \Theta_0^c \\ & \phantom{\text{for }} \psi \in \Theta_0 \end{array}\right\}$$

Meaning :- An unbiased test is more likely to reject H_0 if $\theta \in \Theta_0^c$ over $\theta \in \Theta_0$.

Example 8.3.3 (Revisit) : $X \sim \text{Normal}(\theta, \sigma^2)$ $\theta_0 \in \mathbb{R}$

σ – known $c > 0$ fixed

$X_1 \dots X_n$ i.i.d sample from X

$H_0: \theta \le \theta_0$

$H_1: \theta > \theta_0$

Reject if $\dfrac{\sqrt{n}(\overline{X} - \theta_0)}{\sigma} \geq c$

Power function.

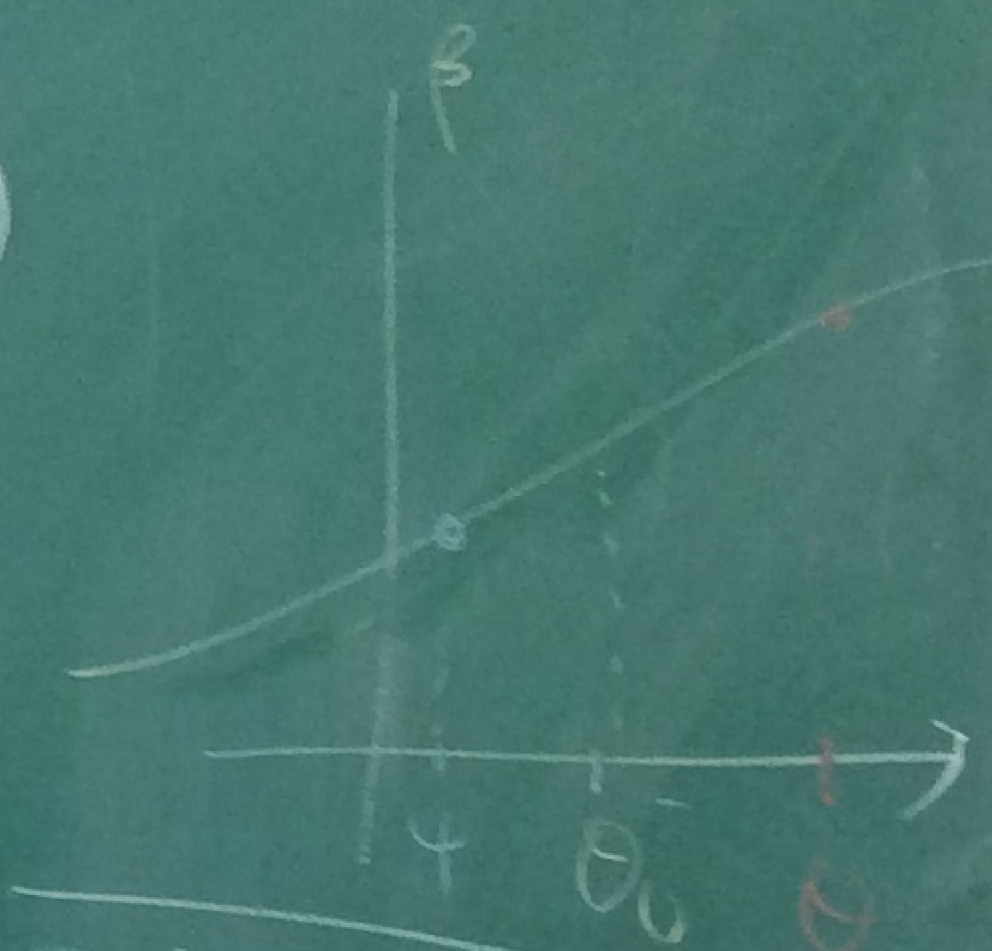
$$\beta(\theta) = \mathbb{P}_\theta\left(\frac{\sqrt{n}(\overline{X} - \theta_0)}{\sigma} > c \right)$$

$$(\text{Check Example}) = \mathbb{P}\left(Z > c + \frac{\theta_0 - \theta}{\sigma/\sqrt{n}} \right) \quad Z \sim \text{Normal}(0,1)$$

observed earlier $\beta(\cdot)$ is an increasing function of θ.

Is Test unbiased?

$$\begin{cases} \theta \in \Theta_0^c \\ \psi \in \Theta_0 \end{cases}$$ check. $\beta(\theta) \geq \beta(\psi)$



$$\begin{cases} \theta \in (\theta_0, \infty) \\ \psi \in (-\infty, \theta_0] \end{cases}$$ is $\beta(\theta) \geq \beta(\psi)$?

As $\psi \leq \theta_0 < \theta$ & $\beta(\cdot)$ is increasing function $\Rightarrow$

$$\begin{cases} \beta(\psi) \leq \beta(\theta_0) \leq \beta(\theta) \\ \Rightarrow \beta(\psi) \leq \beta(\theta) \\ \Rightarrow \beta(\theta) \geq \beta(\psi) \end{cases}$$

IS UNBIASED. □

For a given hypotheses there may be many unbiased tests

§3.2 Most powerful tests

$$\begin{cases} H_0 : \theta \in \Theta_0 \\ H_1 : \theta \in \Theta_0^c \end{cases}$$

We have seen many ways of "judging" a test

- A level α-test controls Probability of Type I error. (Infact level $\alpha \Rightarrow P(\text{Type I error})$ is at most α)

unbiased

– [Another criteria] a "good" test is one which has a small Probability of Type II error if $\theta \in \Theta_0^c$ [infact $\Rightarrow$ a large power function when $\theta \in \Theta_0^c$]

So if one had a level α-test such that it has the Smallest probability of type II error amongst all level α test then we can say that this is "can say this is a good candidate for" "best test" in this class of tests.

For a given hypotheses there may be many unbiased tests

$$\begin{cases} H_0 : \theta \in \Theta_0 \\ H_1 : \theta \in \Theta_0^c \end{cases}$$

§3.2 Most powerful tests

We have seen many ways of "judging" a test

- A level α-test controls Probability of Type I error. (infact level $\alpha \Rightarrow P(\text{Type I error})$ is at most α)

– [Another criterion] a "good" test is one which has a small Probability of Type II error if $\theta \in \Theta_0^c$ [in fact $\Rightarrow$ a large power function when $\theta \in \Theta_0^c$]

So if one had a level α-test such that it has the smallest probability of type II error amongst all level α test then we can say that this is "best test" in this class of tests.

(can say this is a good candidate for)

$H_0: \theta \in \Theta_0$

$H_1: \theta \in \Theta_0^c$

<u>Definition</u> Let $\mathcal{C}$ - class of tests.

A test in class $\mathcal{C}$ with power function $\beta(\cdot)$ is (UMP) uniformly most powerful among class $\mathcal{C}$ tests if

$$\beta(\theta) > \beta'(\theta) \quad \forall\, \theta \in \Theta_0^c$$

& $\beta'(\cdot)$ is a power function of any other test in class $\mathcal{C}$.

<u>Meaning</u>: UMP test among $\mathcal{C}$ is one that has smallest Prob of type II error among $\mathcal{C}$ tests

$\mathcal{C}$ - class of tests will be taken as level α-test.

Warning: In many cases the above may be too stringent & an UMP test may not exists.

Question: How does one identify a UMP test?

There is a very fundamental result called Neyman-Pearson lemma that clearly describes which tests are UMP level α test in the situation where both the null & alternative hypotheses are simple i.e $H_0: \theta=\theta_0$

$H_1: \theta=\theta_1$

Theorem 8.3.12 (Neyman-Pearson lemma) Consider testing

$H_0: \theta = \theta_0$, $H_1: \theta = \theta_1$ using the test

with rejection region R that satisfies :-

① $\begin{cases} x \in R & \text{if } f(x|\theta_1) > k f(x|\theta_0) \\ x \in R^c & \text{if } f(x|\theta_1) < k f(x|\theta_0) \end{cases}$

for some $k > 0$. & $f(\cdot|\theta_i)$ is p.m.f or p.d.f corresponding to θ_i.

② Then

(b)

② $\alpha = P_{\theta_0}(X \in R)$.

Then – ⓐ any test that satisfies ① & ② is a
(Sufficient) UMP-level α-test

ⓑ (necessary) If there is a test satisfying ① & ② with
$k > 0$ then any other UMP level α-test is a size α
test and every UMP level α-test satisfies

① except on A such that $P_{\theta_0}(X \in A) = P_{\theta_1}(X \in A) = 0$

Example 8.3.14 [UMP Binomial test]

$X \sim \text{Binomial}(2, \theta)$

$H_0: \theta = 0.5$

$H_1: \theta = 3/4$

$$f(x|\theta) = \begin{cases} {}^2C_x \, \theta^x (1-\theta)^{2-x} & x = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

Check Neyman-Pearson lemma Conditions: $\boxed{\text{What is } R \text{ \& } \alpha\text{?}}$

(1) $f(x|\theta_1) > k f(x|\theta_0)$ if $x \in R$, $f(x|\theta_1) < k f(x|\theta_0)$ $x \in R^c$

(2) $\alpha = P_{\theta_0}(X \in R)$

Assume $\theta_0 = 0.5$, $\theta_1 = 3/4$

$$\frac{f(0|\theta_1)}{f(0|\theta_0)} = \frac{(1/4)^2}{(1/2)^2} = \frac{1}{4}\ ; \quad \frac{f(1|\theta_1)}{f(1|\theta_0)} = \frac{2(3/4)(1-3/4)}{2(1/2)(1-\frac{1}{2})} = 3/4$$

$$\frac{f(2|\theta_1)}{f(2|\theta_0)} = \frac{(3/4)^2}{(1/2)^2} = 9/4$$

__Test 1__: choose $k: 3/4 < k < \frac{9}{4}$

If NP-lemma are satisfied by test we reject if $X=2$ $(R=\{2\})$ & accept otherwise.

$$\alpha = P_{\theta_0}(X \in R) = f(2|0.5) = 1/4$$

Reject if $X=2$ is a level $1/4$ UMP test